

Convergence of the Critical Planar Ising Interfaces to Hypergeometric SLE

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Abstract

We consider the planar Ising model in rectangle $(\Omega; x^L, x^R, y^R, y^L)$ with alternating boundary condition: \ominus along $(x^L x^R)$ and $(y^R y^L)$, and \oplus along $(x^R y^R)$ and $(y^L x^L)$. We prove that the interface of critical Ising model with this boundary condition converges to the so-called hypergeometric SLE_3 . We obtain this result by considering the scaling limit of the pair of Ising interfaces. The method developed in this paper does not require constructing holomorphic observable and the input is the convergence of the interface with Dobrushin boundary condition. This method could be applied to other lattice models, for instance Loop-Erased Random Walk and Double-Dimer model.

Keywords: Critical Planar Ising, Hypergeometric SLE.

1 Introduction

The Lenz-Ising model is introduced to model the ferromagnetism in statistical mechanics. Due to celebrated work of Chelkak and Smirnov [CS12], it is proved that at the critical temperature, the interface of Ising model is conformally invariant. In particular, the interface of critical Ising model with Dobrushin boundary condition converges to SLE_3 [CDCH⁺14], and the interface of critical Ising model with free boundary condition converges to $\text{SLE}_3(-3/2, -3/2)$ [HK13, Izy15], and the interface for multiply-connected domains [Izy13]. In these cases, the proofs are based on constructing holomorphic observables. In this paper, we study the scaling limit of the critical Ising model with alternating boundary conditions: we consider critical Ising model in a rectangle $(\Omega; x^L, x^R, y^R, y^L)$ with \ominus along $(x^L x^R)$ and $(y^R y^L)$, and \oplus along $(x^R y^R)$ and $(y^L x^L)$, see Figure 1.1. With this boundary condition, on the event that there is a vertical crossing of \ominus , we see that there are two interfaces in the model $(\eta^L; \eta^R)$ where η^L is an interface from x^L to y^L and η^R is an interface from x^R to y^R . In this paper, we study the law of the pair $(\eta^L; \eta^R)$. The scaling limit of η^L is the so-called hypergeometric SLE, denoted by hSLE .

There are two features on the method developed in this paper. First, constructing holomorphic observable is the usual way to prove the convergence of interfaces in the critical lattice model; however, with our method, there is no need to construct new observable. The only input we need is the convergence of the interface with Dobrushin boundary condition. Second, there are many works on multiple SLEs trying to study the scaling limit of interfaces in critical lattice model with alternating boundary conditions, see [Dub07, BBK05, KP16], and their works study the local growth of these interfaces. Whereas, our result is “global”: we prove that the scaling limit of η^L is hSLE_3 as a continuous curve from x^L to y^L .

In this paper, we first study the properties of hSLE in Theorem 1.1. Then we study the possible scaling limit of the pair of the interfaces $(\eta^L; \eta^R)$. We realize that there exists only one possible candidate for the limit of the pair $(\eta^L; \eta^R)$, see Theorem 1.2. By identifying the only possible candidate, we prove the convergence of the pair of the interfaces $(\eta^L; \eta^R)$ in the critical Ising model with alternating boundary condition in Theorem 1.3. In particular, this gives the convergence of η^L to hSLE_3 .

Theorem 1.1. *Fix $\kappa \in (0, 8)$ and $0 < x < y$. Let η be the hSLE_κ in \mathbb{H} from 0 to ∞ with marked points (x, y) . Then the process η is almost surely generated by a continuous transient curve. Moreover, the*

process η enjoys reversibility: the time reversal of η is the hSLE_κ in \mathbb{H} from ∞ to 0 with marked points (y, x) .

Theorem 1.2. Fix a topological rectangle $(\Omega; x^L, x^R, y^R, y^L)$. Let $X_0(\Omega; x^L, x^R, y^R, y^L)$ be the collection of pairs of continuous curves $(\eta^L; \eta^R)$ in Ω such that η^L (resp. η^R) is a continuous curve from x^L to y^L (resp. from x^R to y^R) that does not intersect $(x^R y^R)$ (resp. does not intersect $(y^L x^L)$) and that η^L is to the left of η^R . Fix $\kappa \in (0, 4]$.

- (Uniqueness) There exists a unique probability measure on $X_0(\Omega; x^L, x^R, y^R, y^L)$ with the following property: the conditional law of η^R given η^L is SLE_κ in the connected component of $\Omega \setminus \eta^L$ with $(x^R y^R)$ on the boundary, and the conditional law of η^L given η^R is SLE_κ in the connected component of $\Omega \setminus \eta^R$ with $(y^L x^L)$ on the boundary.
- (Identification) Under this probability measure, the marginal law of η^L is hSLE_κ with marked points (x^R, y^R) and the marginal law of η^R is hSLE_κ with marked points (x^L, y^L) .

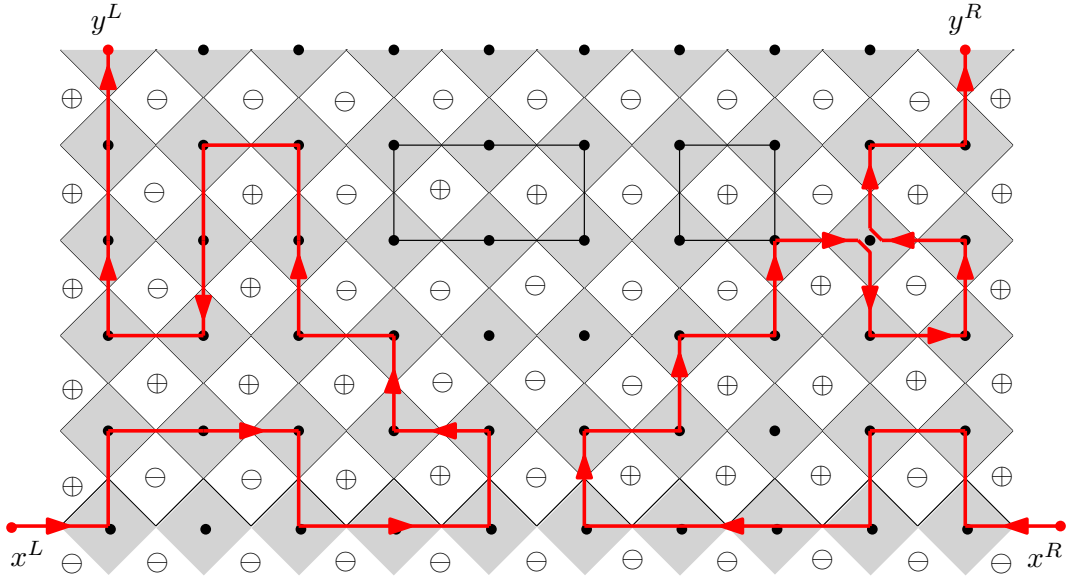


Fig. 1.1: The Ising interface with alternating boundary condition.

Theorem 1.3. Let discrete domains $(\Omega_\delta; x_\delta^L, x_\delta^R, y_\delta^R, y_\delta^L)$ on the square lattice approximate some topological rectangle $(\Omega; x^L, x^R, y^R, y^L)$ as $\delta \rightarrow 0$. Consider the critical Ising model in Ω_δ with the alternating boundary condition: \ominus on $(x^L x^R)$ and $(y^R y^L)$; and \oplus on $(x^R y^R)$ and $(y^L x^L)$. Conditioned on the event that there exists a vertical crossing of \ominus , then there exists a pair of interfaces $(\eta_\delta^L; \eta_\delta^R)$ where η_δ^L (resp. η_δ^R) is the interface connecting x_δ^L to y_δ^L (resp. connecting x_δ^R to y_δ^R). The law of the pair $(\eta_\delta^L; \eta_\delta^R)$ converges weakly to the pair of SLE_κ curves in Theorem 1.2 with $\kappa = 3$ as $\delta \rightarrow 0$. In particular, the law of η_δ^L (resp. η_δ^R) converges weakly to hSLE_3 as $\delta \rightarrow 0$.

A similar conclusion as in Theorem 1.2 also holds for multiple SLE curves; however, it is difficult to identify the marginal law of the curves (except the case with $\kappa = 4$ where the marginal law is $\text{SLE}_4(\rho)$ process, see Remark 4.4). Instead, we could derive the marginal law for the degenerate case: when all the starting points of curves coincide and all the ending points of curves coincide, the marginal law of the curves becomes $\text{SLE}_\kappa(\rho)$ process.

Proposition 1.4. *Fix a Dobrushin domain $(\Omega; x, y)$ and an integer $n \geq 2$. Let $X_0^n(\Omega; x, y)$ be the collection of n non-intersecting curves $(\eta_1; \dots; \eta_n)$ where η_j is a continuous curve in Ω from x to y for $j \in \{1, \dots, n\}$ and that η_j is to the right of η_{j-1} and is to the left of η_{j+1} with the convention that $\eta_0 = (yx)$ and $\eta_{n+1} = (xy)$. Fix $\kappa \in (0, 4]$.*

- (Uniqueness) *There exists a unique probability measure on $X_0^n(\Omega; x, y)$ with the following property: for each $j \in \{1, \dots, n\}$, the conditional law of η_j given η_{j-1} and η_{j+1} is SLE_κ in the region between η_{j-1} and η_{j+1} .*
- (Identification) *Under this probability measure, the marginal law of η_j is $\text{SLE}_\kappa(\rho_j^L; \rho_j^R)$ for $j \in \{1, \dots, n\}$ where $\rho_j^L = 2j - 2$, $\rho_j^R = 2n - 2j$.*

Comparing Theorem 1.2 and Proposition 1.4 with $n = 2$, we see that hSLE_κ degenerates to $\text{SLE}_\kappa(2)$ when the marked points degenerate to the end points, and this is true for all $\kappa \in (0, 8)$, see Remark 3.2. In Theorem 1.2 and Proposition 1.4, we focus on $\kappa \in (0, 4]$. Readers may wonder whether we have similar conclusion for $\kappa \in (4, 8)$. In fact, we believe the conclusion in Theorem 1.2 also holds for $\kappa \in (4, 8)$. However, both of the two parts are not known to our knowledge. The difficulty for the first part is that we can only show the uniqueness when the two curves do not hit each other, see Proposition 4.1. The difficulty in the second part is that we can not show hSLE_κ satisfies the desired property for $\kappa \in (4, 8)$. Whereas, we can still show a weaker version of Theorem 1.2 for the degenerate case when $\kappa \in (4, 8)$, see Lemma 4.5. By applying this result to FK-Ising model, we have the following conclusion.

Proposition 1.5. *Let discrete domains $(\Omega_\delta; x_\delta^L, x_\delta^R, y_\delta^L, y_\delta^R)$ on the square lattice approximate some Dobrushin domain $(\Omega; x, y)$ such that $x_\delta^L, x_\delta^R \rightarrow x$ and $y_\delta^L, y_\delta^R \rightarrow y$ as $\delta \rightarrow 0$. Consider the critical FK-Ising model in Ω_δ with alternating boundary condition: free on (x_δ^L, x_δ^R) and (y_δ^R, y_δ^L) , and wired on (x_δ^R, y_δ^R) and (y_δ^L, x_δ^L) . Conditioned on the event that there are two disjoint vertical dual-crossings, then there exists a pair of interfaces $(\eta_\delta^L; \eta_\delta^R)$ where η_δ^L (resp. η_δ^R) is the interface connecting x_δ^L to y_δ^L (resp. connecting x_δ^R to y_δ^R). The law of $(\eta_\delta^L; \eta_\delta^R)$ converges weakly to the unique pair of curves $(\eta^L; \eta^R)$ in $X_0^2(\Omega; x, y)$ with the following property: Given η^L , the conditional law of η^R is an $\text{SLE}_{16/3}$ conditioned not to hit η^L except at the end points; given η^R , the conditional law of η^L is an $\text{SLE}_{16/3}$ conditioned not to hit η^R except at the end points. In particular, the law of η_δ^L converges weakly to $\text{SLE}_\kappa(\kappa - 2)$ with $\kappa = 16/3$ as $\delta \rightarrow 0$.*

Outline and relation to previous work. We will introduce hypergeometric SLE in Section 3. There were various papers working on variants of hypergeometric SLE with different motivations, see [Zha08, Qia16]. The definitions may be different from ours. So we include a self-contained introduction to hSLE with our motivation in Section 3 and show Theorem 1.1. We will prove Theorem 1.2 and Proposition 1.4 in Section 4. The uniqueness part in Theorem 1.2 was proved in [MS16b, Theorem 4.1] and the identification part was implicitly proved in [Law09, KL07]. We will explain the proof in a self-contained way in Section 4 and give references for omitting details. We will introduce Ising model in Section 5 and prove Theorem 1.3. We will introduce FK-Ising model in Section 6 and prove Proposition 1.5. In [Izy13], the author proved local convergence of Ising interfaces to hSLE_3 by constructing holomorphic observables. The method there also works for multiply connected domains with any number of marked points. We will show Theorem 1.3 by Theorem 1.2 (without constructing any new observable). Our method only requires the input of convergence with Dobrushin boundary condition and it also works for other lattice models, for instance Loop-Erased Random Walk and Double-Dimer model.

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2 Preliminaries

2.1 Space of curves

A planar curve is a continuous mapping from $[0, 1]$ to \mathbb{C} modulo reparameterization. Let X be the set of planar curves. The metric d on X is defined by

$$d(\eta_1, \eta_2) = \inf_{\varphi_1, \varphi_2} \sup_{t \in [0, 1]} |\eta_1(\varphi_1(t)) - \eta_2(\varphi_2(t))|,$$

where the inf is over increasing homeomorphisms $\varphi_1, \varphi_2 : [0, 1] \rightarrow [0, 1]$. The metric space (X, d) is complete and separable. A simple curve is a continuous injective mapping from $[0, 1]$ to \mathbb{C} modulo reparameterization. Let X_{simple} be the subspace of simple curves and denote by X_0 its closure. The curves in X_0 may have multiple points but they do not have self-crossings.

We call $(\Omega; x, y)$ a *Dobrushin domain* if Ω is a non-empty simply connected proper subset of \mathbb{C} and x, y are two distinct boundary points. Denote by (xy) the arc of $\partial\Omega$ from x to y counterclockwise. We say that a sequence of Dobrushin domains $(\Omega_\delta; a_\delta, b_\delta)$ converges to a Dobrushin domain $(\Omega; a, b)$ in the *Carathéodory sense* if $f_\delta \rightarrow f$ uniformly on any compact subset of \mathbb{H} where f_δ (resp. f) is the unique conformal map from \mathbb{H} to Ω_δ (resp. Ω) satisfying $f_\delta(0) = a_\delta, f_\delta(\infty) = b_\delta$ and $f'_\delta(\infty) = 1$ (resp. $f(0) = a, f(\infty) = b, f'(\infty) = 1$).

Given a Dobrushin domain $(\Omega; x, y)$, let $X_{\text{simple}}(\Omega; x, y)$ be the space of simple curves η such that

$$\eta(0) = x, \quad \eta(1) = y, \quad \eta(0, 1) \subset \Omega.$$

Denote by $X_0(\Omega; x, y)$ the closure of $X_{\text{simple}}(\Omega; x, y)$.

We call $(\Omega; a, b, c, d)$ a *quad (or topological rectangle)* if Ω a non-empty simply connected proper subset of \mathbb{C} and a, b, c, d are four distinct boundary points in counterclockwise order. Given a quad $(\Omega; a, b, c, d)$, we denote by $d_\Omega((ab), (cd))$ the extremal distance between (ab) and (cd) in Ω . We say a sequence of quads $(\Omega_\delta; a_\delta, b_\delta, c_\delta, d_\delta)$ converges to a quad $(\Omega; a, b, c, d)$ in the *Carathéodory sense* if $f_\delta \rightarrow f$ uniformly on any compact subset of \mathbb{H} and $\lim_\delta f_\delta^{-1}(b_\delta) = f^{-1}(b)$ and $\lim_\delta f_\delta^{-1}(c_\delta) = f^{-1}(c)$ where f_δ (resp. f) is the unique conformal map from \mathbb{H} to Ω_δ (resp. Ω) satisfying $f_\delta(0) = a_\delta, f_\delta(\infty) = d_\delta$ and $f'_\delta(\infty) = 1$ (resp. $f(0) = a, f(\infty) = d, f'(\infty) = 1$).

Given a quad $(\Omega; x^L, x^R, y^R, y^L)$, let $X_{\text{simple}}(\Omega; x^L, x^R, y^R, y^L)$ be the collection of pairs of simple curves $(\eta^L; \eta^R)$ such that $\eta^L \in X_{\text{simple}}(\Omega; x^L, y^L)$ and $\eta^R \in X_{\text{simple}}(\Omega; x^R, y^R)$ and that $\eta^L \cap \eta^R = \emptyset$. The definition of $X_0(\Omega; x^L, x^R, y^R, y^L)$ is a little bit complicate. Given a quad $(\Omega; x^L, x^R, y^R, y^L)$ and $\epsilon > 0$, let $X_0^\epsilon(\Omega; x^L, x^R, y^R, y^L)$ be the set of pairs of curves $(\eta^L; \eta^R)$ such that

- $\eta^L \in X_0(\Omega; x^L, y^L)$ and $\eta^R \in X_0(\Omega; x^R, y^R)$;
- $d_{\Omega^L}(\eta^L, (x^R y^R)) \geq \epsilon$ where Ω^L the connected component of $\Omega \setminus \eta^L$ with $(x^R y^R)$ on the boundary, and η^R is contained in the closure of Ω^L ;
- $d_{\Omega^R}(\eta^R, (y^L x^L)) \geq \epsilon$ where Ω^R the connected component of $\Omega \setminus \eta^R$ with $(y^L x^L)$ on the boundary, and η^L is contained in the closure of Ω^R .

Define the metric on $X_0^\epsilon(\Omega; x^L, x^R, y^R, y^L)$ by

$$\mathcal{D}((\eta_1^L, \eta_1^R), (\eta_2^L, \eta_2^R)) = \max\{d(\eta_1^L, \eta_2^L), d(\eta_1^R, \eta_2^R)\}.$$

One can check \mathcal{D} is a metric and the space $X_0^\epsilon(\Omega; x^L, x^R, y^R, y^L)$ with \mathcal{D} is complete and separable. Finally, set

$$X_0(\Omega; x^L, x^R, y^R, y^L) = \bigcup_{\epsilon > 0} X_0^\epsilon(\Omega; x^L, x^R, y^R, y^L).$$

Note that $X_0(\Omega; x^L, x^R, y^R, y^L)$ is no longer complete.

Suppose E is a metric space and \mathcal{B}_E is the Borel σ -field. Let \mathcal{P} be the space of probability measures on (E, \mathcal{B}_E) . The Prohorov metric $d_{\mathcal{P}}$ on \mathcal{P} is defined by

$$d_{\mathcal{P}}(\mathbb{P}_1, \mathbb{P}_2) = \inf \{ \epsilon > 0 : \mathbb{P}_1[A] \leq \mathbb{P}_2[A^\epsilon] + \epsilon, \mathbb{P}_2[A] \leq \mathbb{P}_1[A^\epsilon] + \epsilon, \forall A \in \mathcal{B}_E \}.$$

When E is complete and separable, the space \mathcal{P} is complete and separable ([Bil99, Theorem 6.8]); moreover, a sequence \mathbb{P}_n in \mathcal{P} converges weakly to \mathbb{P} if and only if $d_{\mathcal{P}}(\mathbb{P}_n, \mathbb{P}) \rightarrow 0$.

Let Σ be a family of probability measures on (E, \mathcal{B}_E) . We call Σ *relatively compact* if every sequence of elements in Σ contains a weakly convergent subsequence. We call Σ *tight* if, for every $\epsilon > 0$, there exists a compact set K_ϵ such that $\mathbb{P}[K_\epsilon] \geq 1 - \epsilon$ for all $\mathbb{P} \in \Sigma$. By Prohorov's Theorem ([Bil99, Theorem 5.2]), when E is complete and separable, relative compactness is equivalent to tightness.

2.2 Loewner chain

We call a compact subset K of $\overline{\mathbb{H}}$ an \mathbb{H} -*hull* if $\mathbb{H} \setminus K$ is simply connected. Riemann's Mapping Theorem asserts that there exists a unique conformal map g_K from $\mathbb{H} \setminus K$ onto \mathbb{H} such that $\lim_{z \rightarrow \infty} |g_K(z) - z| = 0$. We call such g_K the conformal map from $\mathbb{H} \setminus K$ onto \mathbb{H} normalized at ∞ and we call $a(K) := \lim_{z \rightarrow \infty} z(g_t(z) - z)$ the *half-plane capacity* of K .

Loewner chain is a collection of \mathbb{H} -hulls $(K_t, t \geq 0)$ associated with the family of conformal maps $(g_t, t \geq 0)$ obtained by solving the Loewner equation: for each $z \in \mathbb{H}$,

$$\partial_t g_t(z) = \frac{2}{g_t(z) - W_t}, \quad g_0(z) = z,$$

where $(W_t, t \geq 0)$ is a one-dimensional continuous function which we call the driving function. Let T_z be the *swallowing time* of z defined as $\sup\{t \geq 0 : \min_{s \in [0, t]} |g_s(z) - W_s| > 0\}$. Let $K_t := \overline{\{z \in \mathbb{H} : T_z \leq t\}}$. Then g_t is the unique conformal map from $H_t := \mathbb{H} \setminus K_t$ onto \mathbb{H} normalized at ∞ . Since the half-plane capacity for K_t is $2t$ for all $t \geq 0$, we say that the process $(K_t, t \geq 0)$ is parameterized by the half-plane capacity. We say that $(K_t, t \geq 0)$ can be generated by the continuous curve $(\eta(t), t \geq 0)$ if for any t , the unbounded connected component of $\mathbb{H} \setminus \eta[0, t]$ coincides with $H_t = \mathbb{H} \setminus K_t$.

Here we discuss about the evolution of a point $y \in \mathbb{R}$ under g_t . We assume $y \geq 0$. There are two possibilities: if y is not swallowed by K_t , then we define $Y_t = g_t(y)$; if y is swallowed by K_t , then we define Y_t to be the image of the rightmost of point of $K_t \cap \mathbb{R}$ under g_t . Suppose that $(K_t, t \geq 0)$ is generated by a continuous path $(\eta(t), t \geq 0)$ and that the Lebesgue measure of $\eta[0, \infty] \cap \mathbb{R}$ is zero. Then the process Y_t is uniquely characterized by the following equation:

$$Y_t = y + \int_0^t \frac{2ds}{Y_s - W_s}, \quad Y_t \geq W_t, \quad \forall t \geq 0.$$

In this paper, we may write $g_t(y)$ for the process Y_t .

The convention of driving function can be defined for any simply connected domain via conformal transformation. Fix a Dobrushin domain $(\Omega; x, y)$ and let ϕ be some fixed conformal map from Ω onto \mathbb{H} such that $\phi(x) = 0$ and $\phi(y) = \infty$. Suppose $\eta \in X_{\text{simple}}(\Omega; x, y)$. Then $\phi(\eta)$ is a continuous curve in $X_{\text{simple}}(\mathbb{H}; 0, \infty)$. Thus, if we parameterize $\phi(\eta)$ by its half-plane capacity, then it has a continuous driving process. We use the term the driving process of η in Ω to indicate the driving process in half-plane capacity in \mathbb{H} after the transformation ϕ .

2.3 Convergence of curves

In this section, we first recall the main result of [KS12] and then show a similar result for pairs of curves. We say that a curve η *crosses* Q if there exists a subinterval $[s, t]$ such that $\eta(s, t) \subset Q$ and $\eta[s, t]$ intersects both (ab) and (cd) . Fix a Dobrushin domain $(\Omega; x, y)$, for any curve η in $X_0(\Omega; x, y)$ and any time τ , define

Ω_τ to be the connected component of $\Omega \setminus \eta[0, \tau]$ with y on the boundary. Consider a quad (Q, a, b, c, d) in Ω_τ such that (ab) and (cd) are contained in $\partial\Omega_\tau$. We say that Q is *avoidable* if it does not disconnect $\eta(\tau)$ from y in Ω_τ .

Definition 2.1. A family Σ of probability measures on curves in $X_{\text{simple}}(\Omega; x, y)$ is said to satisfy **Condition C2** if, for any $\epsilon > 0$, there exists a constant $c(\epsilon) > 0$ such that for any $\mathbb{P} \in \Sigma$, any stopping time τ , and any avoidable quad $(Q; a, b, c, d)$ in Ω_τ such that $d_Q((ab), (cd)) \geq c(\epsilon)$, we have

$$\mathbb{P}[\eta[\tau, 1] \text{ crosses } Q \mid \eta[0, \tau]] \leq 1 - \epsilon.$$

Theorem 2.2. [KS12, Corollary 1.7, Proposition 2.6]. Fix a Dobrushin domain $(\Omega; x, y)$. Suppose that $(\eta_n)_{n \in \mathbb{N}}$ is a sequence of curves in $X_{\text{simple}}(\Omega; x, y)$ satisfying Condition C2. Denote by $(W_n(t), t \geq 0)$ the driving process of η_n . Then

- the family $(W_n)_{n \in \mathbb{N}}$ is tight in the metrisable space of continuous functions on $[0, \infty)$ with the topology of uniform convergence on compact subsets of $[0, \infty)$;
- the family $(\eta_n)_{n \in \mathbb{N}}$ is tight in the space of curves X ;
- the family $(\eta_n)_{n \in \mathbb{N}}$, when each curve is parameterized by the half-plane capacity, is tight in the metrisable space of continuous functions on $[0, \infty)$ with the topology of uniform convergence on compact subsets of $[0, \infty)$.

Moreover, if the sequence converges in any of the topologies above it also converges in the two other topologies and the limits agree in the sense that the limiting random curve is driven by the limiting driving function.

Next, we will explain a similar result for pairs of curves. Fix a quad $(\Omega; x^L, x^R, y^R, y^L)$.

Definition 2.3. A family Σ of probability measures on pairs of curves in $X_{\text{simple}}(\Omega; x^L, x^R, y^R, y^L)$ is said to satisfy **Condition C2** if, for any $\epsilon > 0$, there exists a constant $c(\epsilon) > 0$ such that for any $\mathbb{P} \in \Sigma$, the following holds. Given any η^L -stopping time τ^L and any η^R -stopping time τ^R , and any avoidable quad $(Q^R; a^R, b^R, c^R, d^R)$ for η^R in $\Omega \setminus (\eta^L[0, \tau^L] \cup \eta^R[0, \tau^R])$ such that $d_{Q^R}((a^R b^R), (c^R d^R)) \geq c(\epsilon)$, and any avoidable quad $(Q^L; a^L, b^L, c^L, d^L)$ for η^L in $\Omega \setminus (\eta^L[0, \tau^L] \cup \eta^R[0, \tau^R])$ such that $d_{Q^L}((a^L b^L), (c^L d^L)) \geq c(\epsilon)$, we have

$$\mathbb{P}[\eta^R[\tau^R, 1] \text{ crosses } Q^R \mid \eta^L[0, \tau^L], \eta^R[0, \tau^R]] \leq 1 - \epsilon,$$

$$\mathbb{P}[\eta^L[\tau^L, 1] \text{ crosses } Q^L \mid \eta^L[0, \tau^L], \eta^R[0, \tau^R]] \leq 1 - \epsilon.$$

Theorem 2.4. Suppose that $\{(\eta_n^L; \eta_n^R)\}_{n \in \mathbb{N}}$ is a sequence of pairs of curves in $X_{\text{simple}}(\Omega; x^L, x^R, y^R, y^L)$ and denote their laws by $\{\mathbb{P}_n\}_{n \in \mathbb{N}}$. Let Ω_n^L be the connected component of $\Omega \setminus \eta_n^L$ with $(x^R y^R)$ on the boundary and Ω_n^R be the connected component of $\Omega \setminus \eta_n^R$ with $(y^L x^L)$ on the boundary. Define, for each n ,

$$\mathcal{D}_n^L = d_{\Omega_n^L}(\eta_n^L, (x^R y^R)), \quad \mathcal{D}_n^R = d_{\Omega_n^R}(\eta_n^R, (y^L x^L)).$$

Assume that the family $\{(\eta_n^L; \eta_n^R)\}_{n \in \mathbb{N}}$ satisfies Condition C2 and that the sequence of random variables $\{(\mathcal{D}_n^L; \mathcal{D}_n^R)\}_{n \in \mathbb{N}}$ is tight in the following sense: for any $u > 0$, there exists $\epsilon > 0$ such that

$$\mathbb{P}_n[\mathcal{D}_n^L \geq \epsilon, \mathcal{D}_n^R \geq \epsilon] \geq 1 - u, \quad \forall n.$$

Then the sequence $\{(\eta_n^L; \eta_n^R)\}_{n \in \mathbb{N}}$ is relatively compact in $X_0(\Omega; x^L, x^R, y^R, y^L)$.

Proof. By Theorem 2.2, we know that there is subsequence $n_k \rightarrow \infty$ such that $\eta_{n_k}^L$ (resp. $\eta_{n_k}^R$) converges weakly in all three topologies in Theorem 2.2. By Skorohod Represnetation Theorem, we could couple all $(\eta_{n_k}^L; \eta_{n_k}^R)$ in a common space so that $\eta_{n_k}^L \rightarrow \eta^L$ and $\eta_{n_k}^R \rightarrow \eta^R$ almost surely. For $\epsilon > 0$, define

$$K_\epsilon = \{(\eta^L; \eta^R) \in X_{\text{simple}}(\Omega; x^L, x^R, y^R, y^L) : d_{\Omega^L}(\eta^L, (x^R y^R)) \geq \epsilon, d_{\Omega^R}(\eta^R, (y^L x^L)) \geq \epsilon\}.$$

From the assumption, we know that, for any $u > 0$, there exists $\epsilon > 0$ such that $\inf_n \mathbb{P}_n[K_\epsilon] \geq 1 - u$. Therefore, with probability at least $1 - u$, the sequence $(\eta_{n_k}^L; \eta_{n_k}^R)$ converges to $(\eta^L; \eta^R)$ in $X_0^\epsilon(\Omega; x^L, x^R, y^R, y^L) \subset X_0(\Omega; x^L, x^R, y^R, y^L)$. This is true for any $u > 0$, thus we have $(\eta_{n_k}^L; \eta_{n_k}^R)$ converges to $(\eta^L; \eta^R)$ in $X_0(\Omega; x^L, x^R, y^R, y^L)$ almost surely. \square

2.4 SLE and $\text{SLE}_\kappa(\rho)$

Schramm Loewner Evolution SLE_κ is the random Loewner chain $(K_t, t \geq 0)$ driven by $W_t = \sqrt{\kappa} B_t$ where $(B_t, t \geq 0)$ is a standard one-dimensional Brownian motion. In [RS05], the authors prove that $(K_t, t \geq 0)$ is almost surely generated by a continuous transient curve, i.e. there almost surely exists a continuous curve η such that for each $t \geq 0$, H_t is the unbounded connected component of $\mathbb{H} \setminus \eta[0, t]$ and that $\lim_{t \rightarrow \infty} |\eta(t)| = \infty$. There are phase transitions at $\kappa = 4$ and $\kappa = 8$: SLE_κ are simple curves when $\kappa \in [0, 4]$; they have self-touching when $\kappa \in (4, 8)$; and they are space-filling when $\kappa \geq 8$.

It is clear that SLE_κ is scaling invariant, thus we can define SLE_κ in any simply connected domain D from one boundary point x to another boundary point y by the conformal image: let ϕ be a conformal map from \mathbb{H} onto D that sends 0 to x and ∞ to y , then define $\phi(\eta)$ to be SLE_κ in D from x to y . For $\kappa \in (0, 8)$, the curves SLE_κ enjoys *reversibility*: let η be an SLE_κ in D from x to y , then the time-reversal of η has the same law as SLE_κ in D from y to x . The reversibility for $\kappa \in (0, 4]$ was proved in [Zha08], and it was proved for $\kappa \in (4, 8)$ in [MS12].

Hypergeometric functions are defined for $a, b \in \mathbb{R}$, $c \in \mathbb{R} \setminus \{0, -1, -2, -3, \dots\}$ and for $|z| < 1$ by

$${}_2F_1(a, b, c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{n! (c)_n} z^n,$$

where $(q)_n$ is the Pochhammer symbol defined by $(q)_n = 1$ for $n = 0$ and $(q)_n = q(q+1) \cdots (q+n-1)$ for $n \geq 1$. When $c > a + b$, denote by Γ the Gamma function, we have (see [AS92, Equation (15.1.20)]),

$${}_2F_1(a, b, c; 1) = \frac{\Gamma(c) \Gamma(c - a - b)}{\Gamma(c - a) \Gamma(c - b)}.$$

The hypergeometric function is a solution of Euler's hypergeometric differential equation:

$$z(1-z)F''(z) + (c - (a+b+1)z)F'(z) - abF(z) = 0.$$

In this section, we focus on

$$F(z) := {}_2F_1\left(\frac{4}{\kappa}, 1 - \frac{4}{\kappa}, \frac{8}{\kappa}; z\right). \quad (2.1)$$

When $\kappa \in (0, 8)$, we have $F(1) \in (0, \infty)$. In particular, F is smooth for $z \in (-1, 1)$ and is continuous for $z \in (-1, 1]$.

Lemma 2.5. *Fix $\kappa \in (0, 8)$, suppose η is an SLE_κ in \mathbb{H} from 0 to ∞ and $(g_t, t \geq 0)$ is the corresponding family of conformal maps. Fix $0 < x < y$ and let T_x be the swallowing time of x . Define, for $t < T_x$,*

$$J_t = \frac{g'_t(x)g'_t(y)}{(g_t(y) - g_t(x))^2}, \quad Z_t = \frac{g_t(x) - W_t}{g_t(y) - W_t}.$$

Let F be defined through (2.1). Then the following process is a local martingale:

$$M_t := J_t^b Z_t^{2/\kappa} F(Z_t) \mathbb{1}_{\{t < T_x\}}, \quad \text{where } b = \frac{6 - \kappa}{2\kappa}.$$

Proof. By Itô's formula, one can check

$$dJ_t = \frac{-2J_t}{(g_t(y) - W_t)^2} \left(\frac{1}{Z_t} - 1 \right)^2 dt, \quad dZ_t = \frac{(Z_t - 1)dW_t}{g_t(y) - W_t} + \frac{(1 - Z_t)(2 + (2 - \kappa)Z_t)dt}{Z_t(g_t(y) - W_t)^2}.$$

Therefore, the process $J_t^b \psi(Z_t) \mathbb{1}_{\{t < T_x\}}$ is a local martingale if ψ is twice-differentiable and satisfies

$$\kappa z^2(1 - z)\psi''(z) + 2z(2 + (2 - \kappa)z)\psi'(z) - 4b(1 - z)\psi = 0.$$

One can check that $\psi(z) := z^{2/\kappa}F(z)$ satisfies this ODE and hence M_t is a local martingale. Moreover, we have

$$dM_t = M_t \left(\frac{2/\kappa}{W_t - g_t(x)} + \frac{-2/\kappa}{W_t - g_t(y)} - \frac{F'(Z_t)}{F(Z_t)} \left(\frac{1 - Z_t}{g_t(y) - W_t} \right) \right) dW_t.$$

We will show that M_t is actually a uniform integrable martingale in Proposition 3.4. \square

$\text{SLE}_\kappa(\rho)$ processes are variants of SLE_κ where one keeps track of multiple points on the boundary. Suppose $\underline{x}^L = (x^{1,L} < \dots < x^{1,L})$ where $x^{1,L} \leq 0$ and $\underline{x}^R = (x^{1,R} < \dots < x^{r,R})$ where $x^{1,R} \geq 0$ and $\underline{\rho}^L = (\rho^{1,L}, \dots, \rho^{1,L})$, $\underline{\rho}^R = (\rho^{1,R}, \dots, \rho^{r,R})$ where $\rho^{i,q} \in \mathbb{R}$ for $q \in \{L, R\}$ and $i \in \mathbb{N}$. An $\text{SLE}_\kappa(\underline{\rho}^L; \underline{\rho}^R)$ process with force points $(\underline{x}^L; \underline{x}^R)$ is the Loewner evolution driven by W_t which is the solution to the system of integrated SDEs:

$$\begin{aligned} W_t &= \sqrt{\kappa}B_t + \sum_i \int_0^t \frac{\rho^{i,L}ds}{W_s - V_s^{i,L}} + \sum_i \int_0^t \frac{\rho^{i,R}ds}{W_s - V_s^{i,R}}, \\ V_t^{i,q} &= x^{i,q} + \int_0^t \frac{2ds}{V_s^{i,q} - W_s}, \quad \text{for } q \in \{L, R\}, i \in \mathbb{N}, \end{aligned} \tag{2.2}$$

where B_t is one-dimensional Brownian motion. Define the *continuation threshold* to be the infimum of the time t for which

$$\text{either } \sum_{i: V_t^{i,L} = W_t} \rho^{i,L} \leq -2, \quad \text{or} \quad \sum_{i: V_t^{i,R} = W_t} \rho^{i,R} \leq -2.$$

The process is well-defined up to the continuation threshold, and it is generated by continuous curve up to and including the continuation threshold, see [MS16a].

In this paper, we only use $\text{SLE}_\kappa(\rho)$ with two force points: $\text{SLE}_\kappa(\rho^L; \rho^R)$ with force points $(x^L \leq 0 \leq x^R)$ or $\text{SLE}_\kappa(\rho, \nu)$ with force points $(0 \leq x < y)$. To simplify notations, we only discuss properties of these two kinds of processes. The behavior of $\text{SLE}_\kappa(\rho, \nu)$ varies according to different ρ, ν , see [Dub09, Lemma 15]. We list some of them that will be helpful later.

- If $\rho \geq \kappa/2 - 2$, the curve never hits the interval $[x, y)$. If $\rho + \nu \geq \kappa/2 - 2$, the curve never hits the interval $[y, \infty)$.
- If $\rho > -2$ and $\rho + \nu \in (\kappa/2 - 4, \kappa/2 - 2)$, the curve hits the interval (y, ∞) at finite time.
- If $\rho > -2$ and $\rho + \nu \leq \kappa/2 - 4$, the curve accumulates at the point y at finite time.

By Girsanov's Theorem, the law of $\text{SLE}_\kappa(\rho, \nu)$ process can be obtained by weighting the law of ordinary SLE_κ , see [SW05, Theorem 6]:

Lemma 2.6. Fix $\kappa \in (0, 8)$ and $0 < x < y$. Let η be an SLE_κ in \mathbb{H} from 0 to ∞ and $(g_t, t \geq 0)$ be the corresponding family of conformal maps. Define

$$\begin{aligned} M_t(x, y) &= g'_t(x)^{\rho(\rho+4-\kappa)/(4\kappa)} (g_t(x) - W_t)^{\rho/\kappa} \\ &\quad \times g'_t(y)^{\nu(\nu+4-\kappa)/(4\kappa)} (g_t(y) - W_t)^{\nu/\kappa} \times (g_t(y) - g_t(x))^{\rho\nu/(2\kappa)}. \end{aligned}$$

Then $M_t(x, y)$ is a local martingale for SLE_κ and the law of SLE_κ weighted by $M_t(x, y)$ is equal to the law of $\text{SLE}_\kappa(\rho, \nu)$ with force points (x, y) up to the swallowing time of x .

Lemma 2.7. *Fix $\kappa \in (4, 8)$. For $y > 1$, let \mathbb{P}_y be the law of SLE_κ conditioned not to hit the interval $(1, y)$. Then \mathbb{P}_y converges weakly to $\text{SLE}_\kappa(\kappa - 4)$ as $y \rightarrow \infty$.*

Proof. This was proved in [MS16b, Proposition 5.2], we will briefly recall the proof here. Let η be an SLE_κ and denote by \mathbb{P} its law. By Lemma 2.6, we know that the process:

$$M_t := (g_t(1) - W_t)^{(\kappa-4)/\kappa}$$

is a local martingale for η and the law of η weighted by M becomes $\text{SLE}_\kappa(\kappa - 4)$ up to the first time that η swallows 1. Since $\text{SLE}_\kappa(\kappa - 4)$ does not hit the interval $(1, \infty)$, we know that M_t is in fact a uniformly integrable martingale and the law of η weighted by M is $\text{SLE}_\kappa(\kappa - 4)$ for all time. For $N \geq 0$, define $\tau_N = \inf\{t : M_t = N\}$. By Optional Stopping Theorem, we have $\mathbb{E}[M_{\tau_N \wedge \tau_0}] = 1$, thus the event $E_N = \{\tau_N < \tau_0\}$ has probability $1/N$. Therefore, weighting the law of η by M is equivalent to conditioning η on E_N up to $\tau_N \wedge \tau_0$. This is true for all N , thus weighting the law of η by M is equivalent to conditioning η not to hit the interval $(1, \infty)$. \square

2.5 Gaussian Free Field

Suppose that $D \subsetneq \mathbb{C}$ is a proper domain with harmonically non-trivial boundary (i.e. a Brownian motion started at a point in D hits ∂D almost surely.) For $f, g \in L^2(D)$, we denote by (f, g) the inner product of $L^2(D)$: $(f, g) = \int_D f(z)g(z)d^2z$, where d^2z is the Lebesgue area measure. Denote by $H_s(D)$ the space of real-valued smooth functions which are compactly supported in D . This space has a *Dirichlet inner product* defined by

$$(f, g)_\nabla = \frac{1}{2\pi} \int_D \nabla f(z) \cdot \nabla g(z) d^2z.$$

Denote by $H(D)$ the Hilbert space completion of $H_s(D)$.

The *zero-boundary* GFF on D is a random sum of the form $h = \sum_{j=1}^\infty \alpha_j f_j$, where the α_j are i.i.d. one-dimensional standard Gaussians (with mean zero and variance 1) and the f_j are an orthonormal basis for $H(D)$. This sum almost surely diverges within $H(D)$; however, it does converge almost surely in the space of distributions—that is, the limit $\sum_j \alpha_j (f_j, p)$ almost surely exists for all $p \in H_s(D)$, and the limiting values, denoted by (h, p) , as a function of p is almost surely a continuous functional on $H_s(D)$. For any $f \in H_s(D)$, let $p = -\Delta f \in H_s(D)$, and define $(h, f)_\nabla := \frac{1}{2\pi} (h, p)$. Then $(h, f)_\nabla$ is a mean-zero Gaussian with variance

$$\frac{1}{4\pi^2} \sum_j (f_j, p)^2 = \sum_j (f_j, f)_\nabla^2 = (f, f)_\nabla.$$

The zero-boundary GFF on D is the only random distribution on D with the property that, for each $f \in H_s(D)$, the element $(h, f)_\nabla$ is a mean-zero Gaussian with variance $(f, f)_\nabla$. For any harmonic function h_0 on D , we use the phrase GFF with boundary data h_0 to indicate $h = \tilde{h} + h_0$ where \tilde{h} is a zero-boundary GFF.

In this section, we will introduce level lines and flow lines of GFF and list their properties proved in [SS13, MS16a, WW16]. Let $(K_t, t \geq 0)$ be an $\text{SLE}_\kappa(\underline{\rho}^L; \underline{\rho}^R)$ process with force points $(\underline{x}^L; \underline{x}^R)$ where $W, V^{i,q}$ solves (2.2). Let $(g_t, t \geq 0)$ be the corresponding family of conformal maps and set $f_t = g_t - W_t$. Let h_t^0 be the harmonic function on \mathbb{H} with boundary values given by

$$\begin{cases} -\lambda(1 + \sum_0^j \rho^{i,L}), & \text{if } x \in [V_t^{j+1,L}, V_t^{j,L}), \\ \lambda(1 + \sum_0^j \rho^{i,R}), & \text{if } x \in [V_t^{j,R}, V_t^{j+1,R}), \end{cases}$$

where $\lambda = \pi/\sqrt{\kappa}$ with the convention that $\rho^{0,L} = \rho^{0,R} = 0, x^{0,L} = 0_-, x^{l+1,L} = -\infty, x^{0,R} = 0_+, x^{r+1,R} = \infty$. Define

$$h_t(z) = h_t^0(f_t(z)) - \chi \arg f_t'(z), \quad \text{where } \chi = 2/\sqrt{\kappa} - \sqrt{\kappa}/2.$$

There exists a coupling (h, K) where \tilde{h} is a zero-boundary GFF on \mathbb{H} and $h = \tilde{h} + h_0$ such that the following is true. Suppose that τ is any K -stopping time before the continuation threshold. Then the conditional law of h restricted to $\mathbb{H} \setminus K_\tau$ given K_τ is the same as the law of $h_\tau + \tilde{h} \circ f_\tau$. In this coupling, the process K is almost surely determined by h . When $\kappa \in (0, 4)$, we refer to the $\text{SLE}_\kappa(\underline{\rho}^L; \underline{\rho}^R)$ curve in this coupling as the *flow line* of the field h ; and for $\theta \in \mathbb{R}$, we use the phrase flow line of angle θ to indicate the flow line of $h + \theta\chi$. When $\kappa = 4$, we refer to the $\text{SLE}_4(\underline{\rho}^L; \underline{\rho}^R)$ in this coupling as the *level line* of the field h ; and for $u \in \mathbb{R}$, we use the phrase level line with height u to indicate the level line of $h - u$. In this paper, we focus on $\kappa \in (0, 4]$. We usually fix $\kappa \in (0, 4)$ and set $\kappa' = 16/\kappa$. For $\kappa' > 4$, we refer to the $\text{SLE}_{\kappa'}(\underline{\rho}^L; \underline{\rho}^R)$ curve coupled with $-h$ in the coupling as the *counterflow line* of h .

In the rest of this section, we fix the following constants:

$$\kappa \in (0, 4), \quad \kappa' = 16/\kappa, \quad \lambda = \pi/\sqrt{\kappa}, \quad \chi = 2/\sqrt{\kappa} - \sqrt{\kappa}/2. \quad (2.3)$$

The flow lines and counterflow lines of GFF interact in a nice way. Suppose that h is a GFF on \mathbb{H} with piecewise constant data. For $\theta \in \mathbb{R}$, let η_θ be the flow line of h with angle θ . Fix $\theta_1 > \theta_2 > \theta_3$ and suppose that η_{θ_1} and η_{θ_3} do not hit their continuation threshold. Then the flow line η_{θ_2} stays to the left of η_{θ_1} and stays to the right of η_{θ_3} . Moreover, given η_{θ_1} and η_{θ_3} , the conditional law of η_{θ_2} is $\text{SLE}_\kappa(\rho^L; \rho^R)$ where

$$\rho^L = -2 + (\theta_1\chi - \theta_2\chi)/\lambda, \quad \rho^R = -2 + (\theta_2\chi - \theta_3\chi)/\lambda.$$

Suppose that h is a GFF on \mathbb{H} with piecewise constant boundary data. Let η' be the counterflow line of h from ∞ to 0 and assume that the continuation threshold of η' is not hit and η' is nowhere boundary filling. Let η_+ be the flow line of h with angle $\pi/2$ and η_- be the flow line of h with angle $-\pi/2$. Then η_+ is the left boundary of η' and η_- is the right boundary of η' . Combining these facts, we obtain the following decomposition of η' .

Lemma 2.8. *Fix $\kappa \in (2, 4)$ and $\kappa' = 16/\kappa \in (4, 8)$ and $\rho^L > -2, \rho^R > -2$. Let η' be an $\text{SLE}_{\kappa'}(\rho^L; \rho^R)$ in \mathbb{H} from ∞ to 0, and denote by η_+ its left boundary and η_- its right boundary. Then we have the following.*

- The law of η_+ is $\text{SLE}_\kappa(\kappa - 4 + \kappa\rho^L/4; \kappa/2 - 2 + \kappa\rho^R/4)$.
- Given η_- , the conditional law of η_+ is $\text{SLE}_\kappa(\kappa - 4 + \kappa\rho^L/4; -\kappa/2)$.
- Given η_+ , the conditional law of η' is $\text{SLE}_{\kappa'}(\kappa'/2 - 4; \rho^R)$.
- Given η_+ and η_- , the conditional law of η' is $\text{SLE}_{\kappa'}(\kappa'/2 - 4; \kappa'/2 - 4)$.

3 Hypergeometric SLE and Proof of Theorem 1.1

Fix $\kappa \in (0, 8)$ and two boundary points $0 < x < y$. Recall that F is the hypergeometric function defined in (2.1). *Hypergeometric SLE*, denoted by hSLE_κ , with marked points (x, y) is the random Loewner chain driven by W which is the solution to the following system of integrated SDEs:

$$\begin{aligned} W_t &= \sqrt{\kappa}B_t + \int_0^t \frac{2ds}{W_s - V_s^x} + \int_0^t \frac{-2ds}{W_s - V_s^y} - \int_0^t \kappa \frac{F'(Z_s)}{F(Z_s)} \left(\frac{1 - Z_s}{V_s^y - W_s} \right) ds, \\ V_t^x &= x + \int_0^t \frac{2ds}{V_s^x - W_s}, \quad V_t^y = y + \int_0^t \frac{2ds}{V_s^y - W_s}, \quad \text{where } Z_t = \frac{V_t^x - W_t}{V_t^y - W_t}, \end{aligned} \quad (3.1)$$

where B_t is one-dimensional Brownian motion. It is clear that the process is well-defined up to the swallowing time of x . Moreover, by Girsanov's Theorem, one can check that the law of hSLE_κ with marked points (x, y) , up to the swallowing time of x , can be constructed by weighting the law of SLE_κ by the local martingale given in Lemma 2.5.

Proposition 3.1. *Fix $\kappa \in (0, 8)$ and $0 < x < y$. The hSLE_κ in \mathbb{H} from 0 to ∞ with marked points (x, y) is well-defined for all time and it is almost surely generated by a continuous transient curve, and the curve does not hit the interval $[x, y]$.*

Before proving Proposition 3.1, let us compare hSLE_κ with $\text{SLE}_\kappa(\rho, \nu)$ process. By Girsanov's Theorem, one can check that the Radon-Nikodym derivative of the law of hSLE_κ with marked points (x, y) with respect to the law of $\text{SLE}_\kappa(2, \kappa - 6)$ with force points (x, y) is given by

$$R_t = \frac{F(Z_t)}{F(x/y)} \left(\frac{g_t(y) - W_t}{y} \right)^{4/\kappa - 1}, \quad \text{where } Z_t = \frac{g_t(x) - W_t}{g_t(y) - W_t}.$$

Note that $0 \leq Z_t \leq 1$ for all t and $F(z)$ is bounded for $z \in [0, 1]$. Define, for $n \geq 1$,

$$T_y^n = \inf\{t : g_t(y) - W_t \leq 1/n \text{ or } g_t(y) - W_t \geq n\}.$$

Then we see that $R_{T_y^n}$ is bounded. Therefore, the law of hSLE_κ is absolutely continuous with respect to the law of $\text{SLE}_\kappa(2, \kappa - 6)$ up to T_y^n . Since $\text{SLE}_\kappa(2, \kappa - 6)$ is generated by a continuous curve up to T_y and it does not hit the interval $[x, y]$ (since $2 \geq \kappa/2 - 2$ for $\kappa \in (0, 8)$), we know that hSLE_κ is generated by a continuous curve up to T_y^n and it does not hit the interval $[x, y]$ up to T_y^n . Let $n \rightarrow \infty$, we see that hSLE_κ is generated by continuous curve and it does not hit the interval $[x, y]$ up to $T_y = \lim_n T_y^n$.

Remark 3.2. *From the above argument, we see that hSLE_κ with marked points (x, y) converges weakly to $\text{SLE}_\kappa(2)$ with force point x when $y \rightarrow \infty$.*

Note that the absolute continuity of hSLE_κ with respect to $\text{SLE}_\kappa(2, \kappa - 6)$ is not preserved as $n \rightarrow \infty$, since R_t may be no longer bounded away from 0 or ∞ as $t \rightarrow T_y$. The following lemma discusses the behavior of hSLE_κ as $t \rightarrow T_y$.

Lemma 3.3. *The hSLE_κ is well-defined and is generated by continuous curve up to and including the swallowing time of y , denoted by T_y . Moreover, the curve does not hit the interval $[x, y]$; $T_y = \infty$ when $\kappa \leq 4$; and the curve accumulates at a point in the interval (y, ∞) as $t \rightarrow T_y < \infty$ when $\kappa \in (4, 8)$.*

Proof. One can check that $\text{hSLE}_\kappa(K_t, t \geq 0)$ is scaling invariant: for any $\lambda > 0$, the process $(\lambda K_{t/\lambda^2}, t \geq 0)$ has the same law as hSLE_κ with marked points $(\lambda x, \lambda y)$. Thus, we may assume $y = 1$ and $x \in (0, 1)$, and denote T_y by T . In this lemma, we discuss the behavior of hSLE_κ as $t \rightarrow T$ and we will argue that the process does not accumulate at the point 1. To this end, we perform a standard change of coordinate and parameterize the process according the capacity seen from the point 1, see [SW05, Theorem 3] or [Qia16, Section 4.3.3].

Set $f(z) = z/(1 - z)$. Clearly, f is the conformal Möbius transform of \mathbb{H} sending the points $(0, 1, \infty)$ to $(0, \infty, -1)$. Consider the image of $(K_t, 0 \leq t \leq T)$ under f : $(\tilde{K}_s, 0 \leq s \leq \tilde{S})$ where we parameterize this curve by its capacity $s(t)$ seen from ∞ . Let (\tilde{g}_s) be the corresponding family of conformal maps and (\tilde{W}_s) be the driving function. Let f_t be the Möbius transform of \mathbb{H} such that $\tilde{g}_s \circ f = g_t \circ f_t$ where $s = s(t)$. By expanding $\tilde{g}_s = g_t \circ f_t \circ f^{-1}$ around ∞ and comparing the coefficients in both sides, we have

$$f_t(z) = -1 - \frac{g_t''(1)}{2g_t'(1)} + \frac{g_t'(1)}{g_t(1) - z}.$$

Thus, with $s = s(t)$,

$$\tilde{W}_s = f_t(W_t) = -1 - \frac{g_t''(1)}{2g_t'(1)} + \frac{g_t'(1)}{g_t(1) - W_t}, \quad d\tilde{W}_s = \frac{(\kappa - 6)g_t'(1)dt}{(g_t(1) - W_t)^3} + \frac{g_t'(1)dW_t}{(g_t(1) - W_t)^2}.$$

Define

$$\tilde{V}_s^x = f_t(V_t^x), \quad \tilde{V}_s^\infty = f_t(\infty), \quad \tilde{Z}_s = \frac{\tilde{V}_s^x - \tilde{W}_s}{\tilde{V}_s^x - \tilde{V}_s^\infty} = Z_t.$$

Plugging in the time change

$$\dot{s}(t) = f'_t(W_t)^2 = \frac{g'_t(1)^2}{(g_t(1) - W_t)^4},$$

we obtain

$$d\tilde{W}_s = \sqrt{\kappa}d\tilde{B}_s + \frac{2ds}{\tilde{W}_s - \tilde{V}_s^x} + \frac{(\kappa - 6)ds}{\tilde{W}_s - \tilde{V}_s^\infty} - \kappa \frac{F'(\tilde{Z}_s)}{F(\tilde{Z}_s)} \frac{ds}{\tilde{V}_s^x - \tilde{V}_s^\infty},$$

where \tilde{B}_s is one-dimensional Brownian motion. By Girsanov's Theorem, the Radon-Nikodym derivative of the law of \tilde{K} with respect to the law of $\text{SLE}_\kappa(\kappa - 6; 2)$ with force points $(-1; \tilde{x} := x/(1 - x))$ is given by

$$R_s = \frac{F(Z_s)}{F(x)} \left(\frac{g_s(\tilde{x}) - g_s(-1)}{(1 - x)^{-1}} \right)^{-2/\kappa}, \quad \text{where } Z_s = \frac{g_s(\tilde{x}) - W_s}{g_s(\tilde{x}) - g_s(-1)}.$$

Note that $0 \leq Z_s \leq 1$ and $F(z)$ is bounded for $z \in [0, 1]$; and that the process $g_s(\tilde{x}) - g_s(-1)$ is increasing, thus $g_s(\tilde{x}) - g_s(-1) \geq 1/(1 - x)$. Let S be the swallowing time of -1 . Define, for $n \geq 1$,

$$S^n = \inf\{t : K_t \text{ exits } B(0, n)\}.$$

Then R_s is bounded up to $S \wedge S^n$, and thus the process \tilde{K} is absolutely continuous with respect to $\text{SLE}_\kappa(\kappa - 6; 2)$ up to $S \wedge S^n$. We list some properties of $\text{SLE}_\kappa(\kappa - 6; 2)$ with force points $(-1; \tilde{x} = x/(1 - x))$ here: it is generated by continuous curve up to and including the continuation threshold; the curve does not hit the interval $[\tilde{x}, \infty)$, since $2 \geq \kappa/2 - 2$ when $\kappa \in (0, 8)$ when $\kappa \in (0, 4]$, the curve almost surely accumulates at the point -1 , since $\kappa - 6 \leq \kappa/2 - 4$; when $\kappa \in (4, 8)$, the curve hits the interval $(-\infty, -1)$ at finite time almost surely, since $\kappa - 6 \in (-2, \kappa/2 - 2)$. Therefore, the process \tilde{K} is generated by continuous curve up to and including \tilde{S} . This implies that our original hSLE_κ process $(K_t, t \geq 0)$ is generated by continuous curve up to and including T ; moreover, the curve does not hit $[x, y]$ and accumulates at a point in $(y, \infty) \cup \{\infty\}$ as $t \rightarrow T$. \square

Proof of Proposition 3.1. In Lemma 3.3, we have shown that hSLE_κ is well-defined and is generated by continuous curve up to and including T_y . In particular, when $\kappa \leq 4$, since $T_y = \infty$, we obtain the conclusion for this case. It remains to prove the conclusion for $\kappa \in (4, 8)$. In this case, as $t \rightarrow T_y$, we have $V_t^y - W_t \rightarrow 0$ and $Z_t \rightarrow 1$. Note that $F(z)$ remains bounded as $z \rightarrow 1$; and that $F'(z)(1 - z) \rightarrow 0$ as $z \rightarrow 1$, since (see [AS92, Equations (15.2.1), (15.3.3)])

$$F'(z) = \left(\frac{1}{2} - \frac{2}{\kappa} \right) (1 - z)^{8/\kappa - 2} {}_2F_1 \left(\frac{4}{\kappa}, \frac{12}{\kappa} - 1, \frac{8}{\kappa} + 1; z \right) \approx (1 - z)^{8/\kappa - 2}, \quad \text{as } z \rightarrow 1.$$

Combining these, we know that the SDE (3.1) degenerates to $W_t = \sqrt{\kappa}B_t$ for $t \geq T_y$. Therefore, the process is the same as standard SLE_κ for $t \geq T_y$, and hence is generated by continuous transient curve. \square

Proposition 3.4. *Fix $\kappa \in (0, 8)$ and $0 < x < y$. The local martingale defined in Lemma 2.5 is a uniformly integrable martingale for SLE_κ ; and the law of SLE_κ weighted by this martingale is the same as hSLE_κ with marked points (x, y) .*

Proof of Proposition 3.4 and Theorem 1.1. In Lemma 2.5, we have shown that M_t is a local martingale up to the swallowing time of x . Note that J_t is decreasing in t , thus $J_t \leq J_0$. Therefore M_t is bounded as long as J_t and Z_t are bounded from below. Define, for $n \geq 1$,

$$T^n = \inf\{t : J_t \leq 1/n \text{ or } Z_t \leq 1/n\}.$$

Then $M_{t \wedge T^n}$ is a bounded martingale; moreover, the law of SLE_κ weighted by M_t is the law of hSLE_κ up to T^n . By Proposition 3.1, we know that hSLE_κ is generated by a continuous transient curve and the curve never hits the interval $[x, y]$. Therefore, M_t is actually a uniformly integrable martingale for SLE_κ . This completes the proof of Proposition 3.4.

We have shown that hSLE_κ is generated by continuous curve in Proposition 3.1, to show Theorem 1.1, it remains to show the reversibility. To this end, we will derive the explicit formula for M_∞ . Given a deterministic continuous curve η in $\bar{\mathbb{H}}$ from 0 to ∞ with continuous driving function that does not hit the interval $[x, y]$, denote by $D(x, y)$ the connected component of $\mathbb{H} \setminus \eta$ with $[x, y]$ on the boundary. We know that

$$\lim_{t \rightarrow \infty} Z_t = 1, \quad \lim_{t \rightarrow \infty} J_t = J_\infty := \frac{g'(x)g'(y)}{(g(y) - g(x))^2},$$

where g is any conformal map from $D(x, y)$ onto \mathbb{H} . One can check that the quantity J_∞ only depends on the region $D(x, y)$ and does not depend on the choice of conformal map g . In fact, the quantity J_∞ is the so-called Poisson kernel of the region $D(x, y)$. Thus we have almost surely $M_\infty = \lim_{t \rightarrow \infty} M_t = J_\infty^b$. Moreover, the Radon-Nikodym derivative of the law of hSLE_κ with marked points (x, y) with respect to the law of SLE_κ is given by M_∞/M_0 . Combining the reversibility of standard SLE_κ and the conformal invariance of the quantity M_∞/M_0 , we have the reversibility of hSLE_κ . \square

Combining the reversibility of hSLE_κ and Remark 3.2, we obtain the reversibility of $\text{SLE}_\kappa(2)$ process (with the force point next to the starting point) for $\kappa \in (0, 8)$. In fact, we already know the reversibility of general $\text{SLE}_\kappa(\rho)$ processes, it was proved in [Zha10, MS16b, MS12, WW13].

4 Proof of Theorem 1.2

4.1 Proof of Theorem 1.2—Uniqueness

Proposition 4.1. *Fix a quad $(\Omega; x^L, x^R, y^R, y^L)$ and $\kappa \in (0, 8), \rho^L > -2, \rho^R > -2, \nu \geq \kappa/2 - 2$. There exists at most one probability measure on pairs of curves $(\eta^L; \eta^R)$ in $X_0(\Omega; x^L, x^R, y^R, y^L)$ with the following property: the conditional law of η^R given η^L is $\text{SLE}_\kappa(\nu; \rho^R)$ in the connected component of $\Omega \setminus \eta^L$ with (x^R, y^R) on the boundary, and the conditional law of η^L given η^R is $\text{SLE}_\kappa(\rho^L; \nu)$ in the connected component of $\Omega \setminus \eta^R$ with (y^L, x^L) on the boundary.*

This proposition was proved for $\kappa \in (0, 4], \nu = 0$ in [MS16b, Theorem 4.1] and the same proof works as long as the two curves do not hit each other. To be self-contained, we will give a brief proof here and point out why this proof only works when the two curves do not hit.

Before proving this proposition, let us first explain that the conclusion for $\kappa \in (4, 8)$ follows easily from the conclusion for $\kappa \in (0, 4]$. Assume the conclusion in Proposition 4.1 is true for $\kappa \in (0, 4]$. Fix $\kappa' \in (4, 8)$ and set $\kappa = 16/\kappa' \in (2, 4)$. Suppose that $(\eta^L; \eta^R)$ is a pair in $X_0(\Omega; x^L, x^R, y^R, y^L)$ such that the conditional law of η^R given η^L is $\text{SLE}_{\kappa'}(\nu; \rho^R)$ and the conditional law of η^L given η^R is $\text{SLE}_{\kappa'}(\rho^L; \nu)$ where $\rho^L, \rho^R > -2$ and $\nu \geq \kappa'/2 - 2$. Let η_-^L be the right boundary of η^L and let η_+^R be the left boundary of η^R . Since the conditional law of η^R given η_-^L is $\text{SLE}_{\kappa'}(\nu; \rho^R)$, we know that the conditional law of η_+^R given η_-^L is (see Lemma 2.8)

$$\text{SLE}_\kappa(\kappa - 4 + \kappa\nu/4; \kappa/2 - 2 + \kappa\rho^R/4).$$

Similarly, the conditional law of η_-^L given η_+^R is

$$\text{SLE}_\kappa(\kappa/2 - 2 + \kappa\rho^L/4; \kappa - 4 + \kappa\nu/4).$$

Note that

$$\kappa/2 - 2 + \kappa\rho^L/4 > -2, \quad \kappa/2 - 2 + \kappa\rho^R/4 > -2, \quad \kappa - 4 + \kappa\nu/4 \geq \kappa/2 - 2.$$

By the conclusion in Proposition 4.1 for $\kappa \leq 4$, we know that there is at most one probability measure on the pair $(\eta_-^L; \eta_+^R)$. Given the pair $(\eta_-^L; \eta_+^R)$, there is only one way to reconstruct the pair $(\eta^L; \eta^R)$, since the conditional law of η^L given $(\eta_-^L; \eta_+^R)$ is $\text{SLE}_{\kappa'}(\rho^L; \kappa'/2 - 4)$; and the conditional law of η^R given $(\eta_-^L; \eta_+^R)$ is $\text{SLE}_{\kappa'}(\kappa'/2 - 4; \rho^R)$. This implies that the conclusion in Proposition 4.1 holds for $\kappa' \in (4, 8)$.

Proof of Proposition 4.1. From the above argument, we only need to prove the conclusion for $\kappa \in (0, 4]$. Define a Markov chain which transitions from a configuration $(\eta^L; \eta^R)$ to another $(\tilde{\eta}^L; \tilde{\eta}^R)$ in the following way: Given a configuration $(\eta^L; \eta^R)$ of non-intersecting curves in $X_0(\Omega; x^L, x^R, y^R, y^L)$, we pick $q \in \{L, R\}$ uniformly and resample η^q according to the conditional law of η^q given the other one. We will argue that this chain has at most one stationary measure. Suppose that μ is any stationary measure for this chain. Fix $\epsilon > 0$ small, and let μ_ϵ be the measure μ conditioned on $X_0^\epsilon(\Omega; x^L, x^R, y^R, y^L)$. Then μ_ϵ is stationary for the ϵ -Markov chain: the chain is defined the same as before except in each step we resample the path conditioned on $X_0^\epsilon(\Omega; x^L, x^R, y^R, y^L)$. Let Σ^ϵ be the set of all such stationary measures. Clearly, Σ^ϵ is convex.

First, we argue that Σ^ϵ is compact. Suppose that ν_n is a sequence in Σ^ϵ converging weakly to ν , we need to show that ν is also stationary for the ϵ -Markov chain. Suppose that $(\eta_n^L; \eta_n^R)$ has law ν_n and $(\eta^L; \eta^R)$ has law ν . By Skorohod Representation Theorem, we could couple all $(\eta_n^L; \eta_n^R)$ and $(\eta^L; \eta^R)$ in a common space so that $\eta_n^L \rightarrow \eta^L$ and $\eta_n^R \rightarrow \eta^R$ almost surely. Let D_n^R be the connected component of $\Omega \setminus \eta_n^L$ with (x^R, y^R) on the boundary; and D_n^L be the connected component of $\Omega \setminus \eta_n^R$ with (y^L, x^L) on the boundary. Define D^L, D^R for $(\eta^L; \eta^R)$ similarly. For $\delta > 0$ small, let U_δ^R be the open set of points in D^R that has distance at least δ to η^L and define U_δ^L in a similar way. For $q \in \{L, R\}$, let $\mathcal{F}^q = \sigma(\eta_n^q, n \geq 1)$. By the convergence of $\eta_n^q \rightarrow \eta^q$, we know that $U_n^q \subset D^q$ for $q \in \{L, R\}$ for n large enough.

For each n , let h_n^L be the GFF in D_n^L with the boundary value so that its flow line from x^L to y^L is $\text{SLE}_\kappa(\rho^L; \nu)$. Define h_n^R, h^L, h^R analogously. We assume that h^L and h_n^L for all n are coupled together so that, given \mathcal{F}^R , they are conditionally independent. The same is true for h_n^R, h^R . Given \mathcal{F}^R , the total variation distance between the law of h_n^L restricted to U_δ^L and the law of h^L restricted to U_δ^L tends to 0; and similar conclusion also holds for $\mathcal{F}^L, h_n^R, h^R$, see [MS16b, Equation (4.1)]:

$$\lim_{n \rightarrow \infty} \|\mathcal{L}[h_n^L|_{U_\delta^L} | \mathcal{F}^R] - \mathcal{L}[h^L|_{U_\delta^L} | \mathcal{F}^R]\|_{TV} = 0, \quad \lim_{n \rightarrow \infty} \|\mathcal{L}[h_n^R|_{U_\delta^R} | \mathcal{F}^L] - \mathcal{L}[h^R|_{U_\delta^R} | \mathcal{F}^L]\|_{TV} = 0.$$

We will deduce that, given \mathcal{F}^L , the flow line from x^R to y^R generated by h_n^R converges to the one generated by h^R . Fix $\epsilon' > 0$, since $\nu \geq \kappa/2 - 2$, there exists $\delta > 0$ such that, given \mathcal{F}^L , the flow line η^R generated by h^R is contained in U_δ^R with probability at least $1 - \epsilon'$ (This is the part of proof that requires the two curves to be non-intersecting). By the total variation convergence, we could choose n_0 such that, for $n \geq n_0$,

$$\|\mathcal{L}[h_n^R|_{U_\delta^R} | \mathcal{F}^L] - \mathcal{L}[h^R|_{U_\delta^R} | \mathcal{F}^L]\|_{TV} \leq \epsilon'.$$

Since the flow lines are deterministic function of the GFF, the total variation distance between the two flow lines given \mathcal{F}^L is at most $2\epsilon'$. This implies that, given \mathcal{F}^L , the total variation distance between the flow line generated by h_n^R converges to the one generated by h^R . Similar result also holds for $\mathcal{F}^R, h_n^L, h^L$. Since total variation convergence implies weak convergence, we have that the transition kernel for the ϵ -Markov chain is continuous. Therefore, the measure ν is stationary. This completes the proof that Σ^ϵ is compact.

Second, we show that Σ^ϵ is characterized by its extremals. Since Σ^ϵ is compact and the space of probability measures on $X_0^\epsilon(\Omega; x^L, y^L, y^R, x^R)$ is complete and separable, Choquet's Theorem [Phe01, Section 3] implies that μ_ϵ can be uniquely expressed as a superposition of extremals in Σ^ϵ . To show that Σ^ϵ consists of at most one element, it suffices to show that there is only one such extremal in Σ^ϵ . Suppose that $\nu, \tilde{\nu}$ are two distinct extremal elements in Σ^ϵ . Lebesgue's Decomposition Theorem tells that there is a unique decomposition $\nu = \nu_0 + \nu_1$ where ν_0 is absolutely continuous with respect to $\tilde{\nu}$ and ν_1 is singular to $\tilde{\nu}$. If both ν_0, ν_1 are non-zero, then they can be normalized to probability measures in Σ^ϵ , this contradicts that ν is extremal. Therefore, either ν is absolutely continuous with respect to $\tilde{\nu}$ or ν is singular to $\tilde{\nu}$. We could argue that ν can not be absolutely continuous with respect to $\tilde{\nu}$. This is proved in [MS16b, Proof of Theorem 4.1].

Finally, we only need to show that ν and $\tilde{\nu}$ can not be singular. Suppose that we have two initial configurations $(\eta_0^L; \eta_0^R) \sim \nu$ and $(\tilde{\eta}_0^L; \tilde{\eta}_0^R) \sim \tilde{\nu}$ sampled independently. First, we set $\eta_1^L = \eta_0^L$ and $\tilde{\eta}_1^L = \tilde{\eta}_0^L$,

and then, given η_1^L and $\tilde{\eta}_1^L$, we sample η_1^R and $\tilde{\eta}_1^R$ according to the conditional law and couple them to maximize the probability for them to be equal. The fact that this probability is positive is guaranteed by Lemma 4.2. Next, we set $\eta_2^R = \eta_1^R$ and $\tilde{\eta}_2^R = \tilde{\eta}_1^R$, and then, given $\eta_2^R, \tilde{\eta}_2^R$, we sample $\eta_2^L, \tilde{\eta}_2^L$ according to the conditional law and couple them to maximize the probability for them to be equal. Lemma 4.2 guarantees that the probability for $(\eta_2^L; \eta_2^R) = (\tilde{\eta}_2^L; \tilde{\eta}_2^R)$ is positive, which implies that ν and $\tilde{\nu}$ can not be singular. This completes the proof that Σ^ϵ contains at most one element. Since this is true for any $\epsilon > 0$, we know that there is at most one stationary measure for the original Markov chain. \square

Lemma 4.2. *Let $(D; x, y)$ and $(\tilde{D}; x, y)$ be two Dobrushin domains such that $\partial\tilde{D}$ agrees with ∂D in neighborhoods of the arc (xy) . Fix $\kappa > 0, \rho^L > (\kappa/2 - 4) \vee -2$ and $\rho^R > -2$. Let η (resp. $\tilde{\eta}$) be an $\text{SLE}_\kappa(\rho^L; \rho^R)$ in D (resp. \tilde{D}) from x to y with force points $(x_-; x_+)$. Then there exists a coupling $(\eta, \tilde{\eta})$ such that the probability for them to be equal is positive.*

Proof. Proof of [MS16b, Lemma 4.2] and the discussion after the proof there. \square

4.2 Proof of Theorem 1.2—Identification

Suppose K is an \mathbb{H} -hull such that $\text{dist}(0, K) > 0$ and let $H := \mathbb{H} \setminus K$. Let ϕ be the conformal map from H onto \mathbb{H} such that $\phi(0) = 0$ and $\lim_{z \rightarrow \infty} \phi(z)/z = 1$. We wish to compare the law of SLE_κ in \mathbb{H} and the law of SLE_κ in H . Suppose η is an SLE_κ in \mathbb{H} from 0 to ∞ and $(g_t, t \geq 0)$ is the corresponding family of conformal maps. Let T be the first time that η hits K . We will study the law of $\tilde{\eta}(t) = \phi(\eta(t))$ for $t < T$. Define \tilde{g}_t to be the conformal map from $\mathbb{H} \setminus \tilde{\eta}[0, t]$ onto \mathbb{H} normalized at ∞ and let h_t be the conformal map from $\mathbb{H} \setminus g_t(K)$ onto \mathbb{H} such that $h_t \circ g_t = \tilde{g}_t \circ \phi$, see [LSW03, Section 5]. Note that $\tilde{W}_t = h_t(W_t)$ is the driving function for $\tilde{\eta}$ and

$$d\tilde{W}_t = h'_t(W_t)dW_t + (\kappa/2 - 3)h''_t(W_t)dt. \quad (4.1)$$

Schwarzian derivative for a conformal map f is defined to be

$$Sf(z) = \frac{f'''(z)}{f'(z)} - \frac{3f''(z)^2}{2f'(z)^2}.$$

Lemma 4.3. *Fix $\kappa \in (0, 4]$. The following process is a uniformly integrable martingale:*

$$M_t := \mathbb{1}_{\{t < T\}} h'_t(W_t)^b \exp\left(-c \int_0^t \frac{Sh_s(W_s)}{6} ds\right), \quad \text{where } b = \frac{6 - \kappa}{2\kappa}, \quad c = \frac{(3\kappa - 8)(6 - \kappa)}{2\kappa}.$$

Moreover, the law of η weighted by M is the same as the law of SLE_κ in H from 0 to ∞ .

Proof. One can check by Itô's Formula that M is a local martingale (see the calculation in [LSW03, Section 5]) and we have

$$dM_t = bM_t \frac{h''_t(W_t)}{h'_t(W_t)} dW_t.$$

Let $\tilde{\mathbb{P}}$ denote the law of η weighted by M . Suppose that B is the Brownian motion in \mathbb{P} and $W_t = \sqrt{\kappa}B_t$, then

$$\tilde{B}_t = B_t + \int_0^t (\kappa/2 - 3) \frac{h''_s(W_s)}{h'_s(W_s)} ds$$

is the Brownian motion under $\tilde{\mathbb{P}}$. Compare this with (4.1), we see that $d\tilde{W}_t = \sqrt{\kappa}h'_t(W_t)d\tilde{B}_t$. Let $s = s(t)$ be the capacity of $\tilde{\eta}[0, t]$, then we have $\dot{s}(t) = h'_t(W_t)^2$. Thus, after the time change, we have $d\tilde{W}_s = \sqrt{\kappa}d\tilde{B}_s$. This implies that the law of η weighted by M is the same as the law of SLE_κ in H up to the first time that η hits K . At the same time, we know that SLE_κ in H will never hits the boundary (except at the end points) when $\kappa \in (0, 4]$. Hence, M is actually a uniformly integrable martingale and the law of η weighted by M is the same as the law of SLE_κ in H for all time. \square

Consider the martingale in Lemma 4.3, since $h'_t(W_t) \rightarrow 1$ as $t \rightarrow \infty$ on $\{T = \infty\}$, we have

$$M_\infty := \lim_{t \rightarrow \infty} M_t = \mathbb{1}_{\{T=\infty\}} \exp \left(-c \int_0^\infty \frac{Sh_s(W_s)}{6} ds \right).$$

We will discuss the quantity in the right hand side. The *Brownian loop measure* is the measure on unrooted Brownian loops. Since we do not need to do calculation with it, we omit the introduction to Brownian loop measure here and refer [LW04, Sections 3,4] for a clear definition. Given a non-empty simply connected domain $\Omega \subsetneq \mathbb{C}$ and two disjoint subsets V_1, V_2 , denote by $m(\Omega; V_1, V_2)$ the Brownian loop measure of loops in Ω that intersect both V_1 and V_2 . This quantity is conformal invariant: $m(f(\Omega); f(V_1), f(V_2)) = m(\Omega; V_1, V_2)$ for any conformal transformation f on Ω . When both of V_1, V_2 are closed, one of them is compact and $\text{dist}(V_1, V_2) > 0$, we have $0 < m(\Omega; V_1, V_2) < \infty$. It is proved in [LW04, Equation (22)] that $-(1/6) \int_0^t Sh_s(W_s) ds = m(\mathbb{H}; K, \eta[0, t])$. Thus we have

$$M_\infty = \mathbb{1}_{\{T=\infty\}} \exp \left(-c \int_0^\infty \frac{Sh_s(W_s)}{6} ds \right) = \mathbb{1}_{\{T=\infty\}} \exp(cm(\mathbb{H}; K, \eta)). \quad (4.2)$$

Proof of Theorem 1.2, Identification. There are three steps: first, we construct a probability measure \mathbb{P} on pairs of curves; second, we show that this probability measure has the property in “Uniqueness”; last, we show that the marginal law of η^L under \mathbb{P} is hSLE_κ .

First, we will construct a probability measure on $(\eta^L; \eta^R)$. Fix $\kappa \in (0, 4]$ and $0 < x < y$. By conformal invariance, it is sufficient to give the construction for the quad $(\mathbb{H}; 0, x, y, \infty)$. Denote by \mathbb{P}_L the law of SLE_κ in \mathbb{H} from 0 to ∞ and denote by \mathbb{P}_R the law of SLE_κ in \mathbb{H} from x to y . Define the measure μ on $X_0(\mathbb{H}; 0, x, y, \infty)$ by

$$\mu[d\eta^L, d\eta^R] = \mathbb{1}_{\{\eta^L \cap \eta^R = \emptyset\}} \exp(cm(\mathbb{H}; \eta^L, \eta^R)) \mathbb{P}_L[d\eta^L] \otimes \mathbb{P}_R[d\eta^R].$$

We argue that the total mass of μ , denoted by $|\mu|$, is finite. Given $\eta^L \in X_0(\mathbb{H}; 0, \infty)$, let g be any conformal map from the connected component of $\mathbb{H} \setminus \eta^L$ with (xy) on the boundary onto \mathbb{H} , then we have

$$\begin{aligned} |\mu| &= \mathbb{E}_L \otimes \mathbb{E}_R \left[\mathbb{1}_{\{\eta^L \cap \eta^R = \emptyset\}} \exp(cm(\mathbb{H}; \eta^L, \eta^R)) \right] \\ &= \mathbb{E}_L \left[\left(\frac{g'(x)g'(y)}{(g(x) - g(y))^2} \right)^b \right] \quad (\text{By Lemma 4.3 and (4.2)}) \\ &= (x - y)^{-2b} (x/y)^{2/\kappa} F(x/y). \quad (\text{By Proposition 3.4}) \end{aligned}$$

This implies that $|\mu|$ is positive finite. We define the probability measure \mathbb{P} to be $\mu/|\mu|$.

Second, we show that, under \mathbb{P} , the conditional law of η^R given η^L is SLE_κ . By the symmetry in the definition of \mathbb{P} , we know that the conditional law of η^L given η^R is also SLE_κ and hence \mathbb{P} satisfies the property in “Uniqueness”. Given η^L , denote by H the connected component of $\mathbb{H} \setminus \eta^L$ with (xy) on the boundary and let g be any conformal map from H onto \mathbb{H} . Denote by \mathbb{P}_R the law of SLE_κ in \mathbb{H} from x to y and by $\tilde{\mathbb{P}}_R$ the law of SLE_κ in H from x to y . By Lemma 4.3, for any bounded continuous function \mathcal{F} on continuous curves, we have

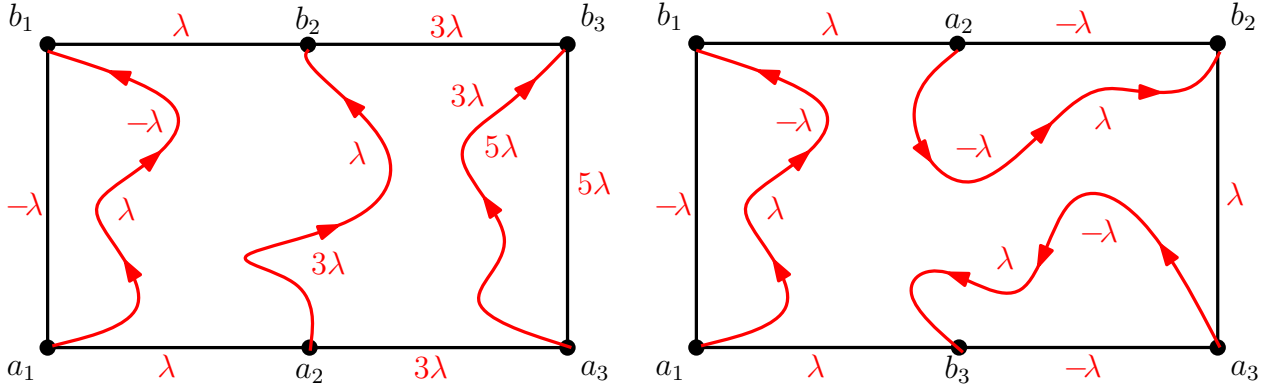
$$\begin{aligned} \mathbb{E}[\mathcal{F}(\eta^R) | \eta^L] &= |\mu|^{-1} \mathbb{E}_R \left[\mathbb{1}_{\{\eta^L \cap \eta^R = \emptyset\}} \exp(cm(\mathbb{H}; \eta^L, \eta^R)) \mathcal{F}(\eta^R) \right] \\ &= |\mu|^{-1} \left(\frac{g'(x)g'(y)}{(g(x) - g(y))^2} \right)^b \tilde{\mathbb{E}}_R[\mathcal{F}(\eta^R)]. \end{aligned}$$

This implies that the conditional law of η^R given η^L is SLE_κ in H .

Finally, we show that, under \mathbb{P} , the marginal law of η^L is hSLE_κ . In fact, the above equation implies that the law of η^L is the law of SLE_κ in \mathbb{H} from 0 to ∞ weighted by

$$\left(\frac{g'(x)g'(y)}{(g(x) - g(y))^2} \right)^b.$$

Therefore, by Proposition 3.4 and the argument in its proof, we see that the law of η^L coincides with hSLE_κ as desired. \square



- (a) The marginal law of η_1 is $\text{SLE}_4(2, 2, -2, -2)$ with force points (a_2, a_3, b_3, b_2) ; the marginal law of η_2 is $\text{SLE}_4(-2, 2; 2, -2)$ with force points $(b_1, a_1; a_3, b_3)$.
- (b) The marginal law of η_1 is $\text{SLE}_4(-2, 2, -2, 2)$ with force points (b_3, a_3, b_2, a_2) conditioned on the event that it accumulates at b_1 . The laws for η_2, η_3 are similar.

Fig. 4.1: Three level lines in GFF.

Remark 4.4. [Degenerate case in Theorem 1.2—level lines of GFF] When $\kappa = 4$, the SDE (3.1) degenerates to the SDE (2.2). In other words, hSLE_4 in \mathbb{H} with marked points (x, y) is the same as $\text{SLE}_4(2, -2)$ with force points (x, y) . In this degenerate case, the pair of curves in Theorem 1.2 can be realized by a pair of level lines of GFF. Fix a quad $(\Omega; x^L, y^L, y^R, x^R)$. Let h be the GFF on Ω with the following boundary data: $-\lambda$ on $(y^L x^L)$, λ on $(x^L x^R) \cup (y^R y^L)$, and 3λ on $(x^R y^R)$. Let η^L be the level line of h with height zero starting from x^L and let η^R be the level line of h with height 2λ starting from x^R . Then we have that the marginal law of η^L is $\text{SLE}_4(2, -2)$ with force points (x^R, y^R) and the conditional law of η^L given η^R is SLE_4 ; the marginal law of η^R is $\text{SLE}_4(2, -2)$ with force points (x^L, y^L) and the conditional law of η^R given η^L is SLE_4 . Thus the pair $(\eta^L; \eta^R)$ is the unique pair described in Theorem 1.2 for $\kappa = 4$.

Moreover, the case with $\kappa = 4$ can be easily generalized to multiple curves. For $n \geq 2$, fix a simply connected domain Ω and $2n$ distinct boundary points $a_1, \dots, a_n, b_n, \dots, b_1$ in counterclockwise order. Let $X_0(\Omega; a_1, \dots, a_n, b_n, \dots, b_1)$ be the collection of non-intersecting curves $(\eta_1; \dots; \eta_n)$ where each η_j is a continuous curve in Ω from a_j to b_j . Then there exists a unique probability measure on $X_0(\Omega; a_1, \dots, a_n, b_n, \dots, b_1)$ such that the conditional law of η_j given η_{j-1} and η_{j+1} is SLE_4 for $j \in \{1, \dots, n\}$ with the convention that $\eta_0 = (b_1 a_1)$ and $\eta_{n+1} = (a_n b_n)$. Under this measure, the marginal law of η_j is $\text{SLE}_4(\rho_j^L; \rho_j^R)$ process where each of $\{a_1, \dots, a_{j-1}, a_{j+1}, \dots, a_n\}$ corresponds to a force point with weight 2, each of $\{b_1, \dots, b_{j-1}, b_{j+1}, \dots, b_n\}$ corresponds to a force point with weight -2 . A similar result also holds for other patterns of multiple curves, see Figure 4.1.

4.3 Proof of Proposition 1.4

Lemma 4.5. *Fix a Dobrushin domain $(\Omega; x, y)$ and $\kappa \in (0, 8)$, $\rho^L > -2$, $\rho^R > -2$ and $\nu \geq \kappa/2 - 2$. There exists a unique probability measure on pairs of curves $(\eta^L; \eta^R)$ in $X_0^2(\Omega; x, y)$ with the following property: the conditional law of η^R given η^L is $\text{SLE}_\kappa(\nu; \rho^R)$ and the conditional law of η^L given η^R is $\text{SLE}_\kappa(\rho^L; \nu)$. Under this probability measure, the marginal law of η^L is $\text{SLE}_\kappa(\rho^L; \rho^R + \nu + 2)$ and the marginal law of η^R is $\text{SLE}_\kappa(\rho^L + \nu + 2; \rho^R)$.*

Proof. By the argument before the proof of Proposition 4.1, we only need to show the conclusion for $\kappa \in (0, 4]$. We first show the existence of the pair by checking that a certain pair of flow lines in GFF satisfies the desired property. Suppose h is a GFF in \mathbb{H} with boundary data

$$-\lambda(2 + \rho^L + \nu/2) \text{ on } (-\infty, 0), \quad \lambda(2 + \rho^R + \nu/2) \text{ on } (0, \infty).$$

Set $\theta = \lambda(\nu/2 + 1)/\chi > 0$. Let η^L be the flow line of h with angle θ and η^R be the flow line of h with angle $-\theta$. Then one can check that the conditional law of η^R given η^L is $\text{SLE}_\kappa(\nu; \rho^R)$ and the conditional law of η^L given η^R is $\text{SLE}_\kappa(\rho^L; \nu)$; and that the marginal laws of η^L and η^R are the same as the one in the statement. It remains to show the uniqueness. This can be obtained by reducing to the setting that the end points are distinct—Proposition 4.1. This is explained at the beginning of [MS16b, Proof of Theorem 4.1]. \square

Proof of Proposition 1.4. We first check that the curves $(\eta_1; \dots; \eta_n)$ can be realized by flow lines of GFF. Suppose h is a GFF on \mathbb{H} with the boundary data: $-n\lambda$ on \mathbb{R}_- and $n\lambda$ on \mathbb{R}_+ . Set $\theta_j = (n+1-2j)\lambda/\chi$ for $j = 1, \dots, n$. Let η_j be the flow line of h with angle θ_j . Then one can check that the conditional law of η_j given η_{j-1} and η_{j+1} is SLE_κ for $j \in \{1, \dots, n\}$ and the marginal law of η_j is $\text{SLE}_\kappa(2j-2; 2n-2j)$.

We prove the uniqueness by induction on n . Lemma 4.5 implies the conclusion for $n = 2$. Suppose the conclusion holds for $n-1$ path and consider $(\eta_1; \dots; \eta_n)$. Applying the hypothesis to $(\eta_2; \dots; \eta_n)$, we know the conditional law of $(\eta_2; \dots; \eta_n)$ given η_1 . In particular, we know that the conditional law of η_2 given η_1 is $\text{SLE}_\kappa(2n-4)$ and the conditional law of η_1 given η_2 is SLE_κ . Thus, by Lemma 4.5, we know that the marginal law of η_1 is $\text{SLE}_\kappa(2n-2)$. The marginal law of η_1 and the conditional law of $(\eta_2; \dots; \eta_n)$ given η_1 uniquely determine the law of $(\eta_1; \dots; \eta_n)$. This completes the proof. \square

Remark 4.6. *The conclusions in Proposition 1.4 hold as long as the terminal points of curves coincide. Suppose that a_1, \dots, a_n, b are boundary points of Ω in counterclockwise order and denote by $X_0^n(\Omega; a_1, \dots, a_n, b)$ the collection of curves $(\eta_1; \dots; \eta_n)$ where $\eta_j \in X_0(\Omega; a_j, b)$ and it is to the right of η_{j-1} and is to the left of η_{j+1} for $j \in \{1, \dots, n\}$ with the convention that $\eta_0 = (ba_1)$ and $\eta_{n+1} = (a_nb)$. Fix $\kappa \in (0, 4]$. Then there is a unique probability measure on $X_0^n(\Omega; a_1, \dots, a_n, b)$ such that the conditional law of η_j given η_{j-1} and η_{j+1} is SLE_κ for $j \in \{1, \dots, n\}$. Under this measure, the marginal law of η_j is $\text{SLE}_\kappa(\rho_j^L; \rho_j^R)$ where each of $\{a_1, \dots, a_{j-1}, a_{j+1}, \dots, a_n\}$ corresponds to a force point with weight 2.*

5 Ising Model and Proof of Theorem 1.3

Notations and terminologies. We focus on the square lattice \mathbb{Z}^2 . Two vertices $x = (x_1, x_2)$ and $y = (y_1, y_2)$ are neighbors if $|x_1 - y_1| + |x_2 - y_2| = 1$, and we write $x \sim y$. The *dual square lattice* $(\mathbb{Z}^2)^*$ is the dual graph of \mathbb{Z}^2 . The vertex set is $(1/2, 1/2) + \mathbb{Z}^2$ and the edges are given by nearest neighbors. The vertices and edges of $(\mathbb{Z}^2)^*$ are called dual-vertices and dual-edges. In particular, for each edge e of \mathbb{Z}^2 , it is associated to a dual edge, denoted by e^* , that it crosses e in the middle. For a finite subgraph G , we define G^* to be the subgraph of $(\mathbb{Z}^2)^*$ with edge-set $E(G^*) = \{e^* : e \in E(G)\}$ and vertex set given by the end-points of these dual-edges. The *medial lattice* $(\mathbb{Z}^2)^\diamond$ is the graph with the centers of edges of \mathbb{Z}^2 as vertex set, and edges connecting nearest vertices. This lattice is a rotated and rescaled version of \mathbb{Z}^2 , see Figure 5.1. The vertices and edges of $(\mathbb{Z}^2)^\diamond$ are called medial-vertices and medial-edges. We identify the faces of $(\mathbb{Z}^2)^\diamond$ with the vertices of \mathbb{Z}^2 and $(\mathbb{Z}^2)^*$. A face of $(\mathbb{Z}^2)^\diamond$ is said to be black if it corresponds to a vertex of \mathbb{Z}^2 and white if it corresponds to a vertex of $(\mathbb{Z}^2)^*$.

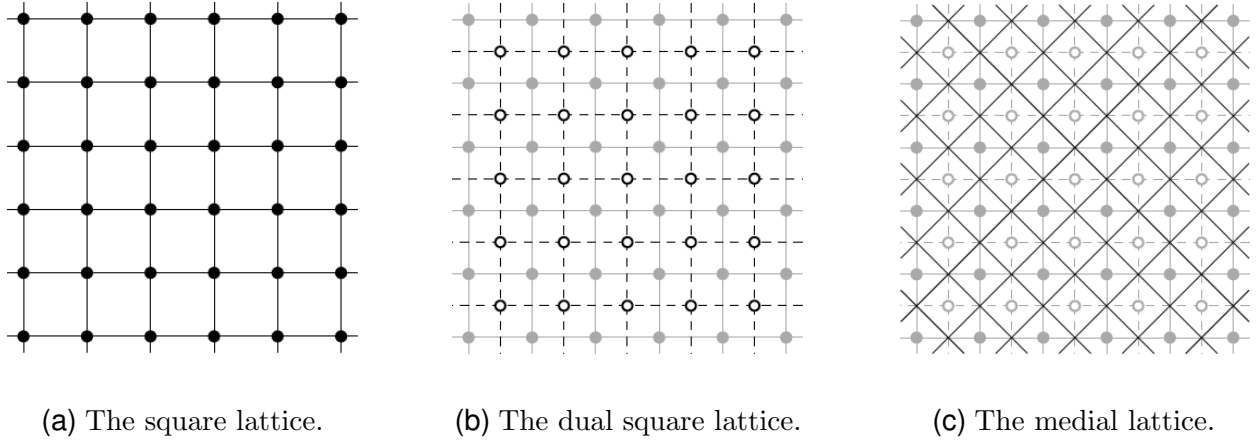


Fig. 5.1: The lattices.

5.1 Ising model

Let Ω be a finite subset of \mathbb{Z}^2 . The Ising model with free boundary conditions is a random assignment $\sigma \in \{\ominus, \oplus\}^\Omega$ of spins $\sigma_x \in \{\ominus, \oplus\}$, where σ_x denotes the spin at the vertex x . The Hamiltonian of the Ising model is defined by

$$H_\Omega^{\text{free}}(\sigma) = - \sum_{x \sim y} \sigma_x \sigma_y.$$

The Ising measure is the Boltzmann measure with Hamiltonian H_Ω^{free} and inverse-temperature $\beta > 0$:

$$\mu_{\beta, \Omega}^{\text{free}}[\sigma] = \frac{\exp(-\beta H_\Omega^{\text{free}}(\sigma))}{Z_{\beta, \Omega}^{\text{free}}}, \quad \text{where } Z_{\beta, \Omega}^{\text{free}} = \sum_{\sigma} \exp(-\beta H_\Omega^{\text{free}}(\sigma)).$$

For a graph Ω and $\tau \in \{\ominus, \oplus\}^{\mathbb{Z}^2}$, one may also define the Ising model with boundary conditions τ by the Hamiltonian

$$H_\Omega^\tau(\sigma) = - \sum_{x \sim y, \{x, y\} \cap \Omega \neq \emptyset} \sigma_x \sigma_y, \quad \text{if } \sigma_x = \tau_x, \forall x \notin \Omega.$$

Ising model satisfies Domain Markov property: Let $\Omega \subset \Omega'$ be two finite subsets of \mathbb{Z}^2 . Let $\tau \in \{\ominus, \oplus\}^{\mathbb{Z}^2}$ and $\beta > 0$. Let X be a random variable which is measurable with respect to vertices in Ω . Then we have

$$\mu_{\beta, \Omega'}^\tau[X \mid \sigma_x = \tau_x, \forall x \in \Omega' \setminus \Omega] = \mu_{\beta, \Omega}^\tau[X].$$

Suppose that $(\Omega; a, b)$ is a Dobrushin domain. The *Dobrushin boundary condition* is the following: \oplus along (ab) , and \ominus along (ba) . This boundary condition is also called domain-wall boundary condition. Suppose that $(\Omega; a, b, c, d)$ is a quad. The *alternating boundary condition* is the following: \oplus along (ab) and (cd) , and \ominus along (bc) and (da) .

The set $\{\ominus, \oplus\}^\Omega$ is equipped with a partial order: $\sigma \leq \sigma'$ if $\sigma_x \leq \sigma'_x$ for all $x \in \Omega$. A random variable X is increasing if $\sigma \leq \sigma'$ implies $X(\sigma) \leq X(\sigma')$. An event \mathcal{A} is increasing if $\mathbb{1}_{\mathcal{A}}$ is increasing. The Ising model satisfies FKG inequality: Let Ω be a finite subset and τ be boundary conditions, and $\beta > 0$. For any two increasing events \mathcal{A} and \mathcal{B} , we have

$$\mu_{\beta, \Omega}^\tau[\mathcal{A} \cap \mathcal{B}] \geq \mu_{\beta, \Omega}^\tau[\mathcal{A}] \mu_{\beta, \Omega}^\tau[\mathcal{B}].$$

As a consequence of FKG inequality, we have the comparison between boundary conditions: For boundary conditions $\tau_1 \leq \tau_2$ and an increasing event \mathcal{A} , we have

$$\mu_{\beta, \Omega}^{\tau_1}[\mathcal{A}] \leq \mu_{\beta, \Omega}^{\tau_2}[\mathcal{A}]. \quad (5.1)$$

The critical value of β in Ising model is given by:

$$\beta_c = \frac{1}{2} \log(1 + \sqrt{2}).$$

The critical Ising model is conformal invariant in the scaling limit. We will list two special properties for the critical Ising model that will be useful later: strong RSW and the convergence of the interface with Dobrushin boundary condition.

Given a quad $(Q; a, b, c, d)$ on the square lattice, we denote by $d_Q((ab), (cd))$ the discrete external distance between (ab) and (cd) in Q , see [Che16, Section 6]. The discrete extremal distance is uniformly comparable to and converges to its continuous counterpart—the classical extremal distance. The quad $(Q; a, b, c, d)$ is crossed by \oplus in an Ising configuration σ if there exists a path of \oplus going from (ab) to (cd) in Q . We denote this event by $(ab) \xleftrightarrow{\oplus} (cd)$.

Proposition 5.1. [CDCH16, Corollary 1.7]. *For each $L > 0$ there exists $c(L) > 0$ such that the following holds: for any quad $(Q; a, b, c, d)$ with $d_Q((ab), (cd)) \geq L$,*

$$\mu_{\beta_c, Q}^{\text{mixed}} \left[(ab) \xleftrightarrow{\oplus} (cd) \right] \leq 1 - c(L),$$

where the boundary conditions are free on $(ab) \cup (cd)$ and \ominus on $(bc) \cup (da)$.

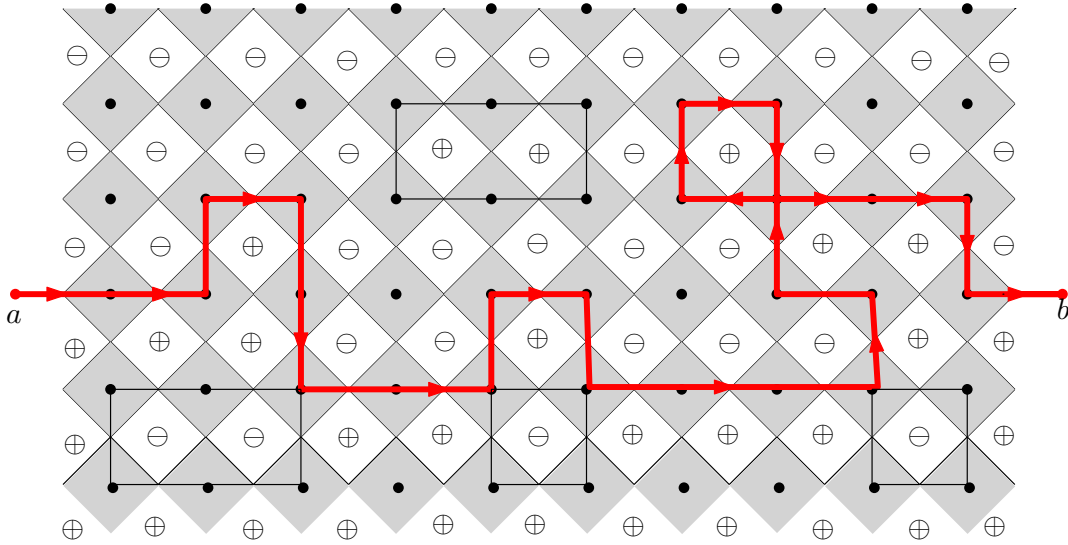


Fig. 5.2: The Ising interface with Dobrushin boundary condition.

For $\delta > 0$, we consider the rescaled square lattice $\delta\mathbb{Z}^2$. The definitions of dual lattice, medial lattice and Dobrushin domains extend to this context, and they will be denoted by $(\Omega_\delta, a_\delta, b_\delta)$, $(\Omega_\delta^*, a_\delta^*, b_\delta^*)$, $(\Omega_\delta^\diamond, a_\delta^\diamond, b_\delta^\diamond)$ respectively. Consider the critical Ising model on $(\Omega_\delta^*, a_\delta^*, b_\delta^*)$. The boundary $\partial\Omega_\delta^*$ is divided into two parts $(a_\delta^* b_\delta^*)$ and $(b_\delta^* a_\delta^*)$. We fix the Dobrushin boundary conditions: \ominus on $(b_\delta^* a_\delta^*)$ and \oplus on $(a_\delta^* b_\delta^*)$. Define the *interface* as follows. It starts from a_δ^\diamond , lies on the primal lattice and turns at every vertex of Ω_δ is such a way that it has always dual vertices with spin \ominus on its left and \oplus on its right. If there is an indetermination when arriving at a vertex (this may happen on the square lattice), turn left. See Figure 5.2. We have the convergence of the interface:

Theorem 5.2. [CDCH⁺14]. *Let $(\Omega_\delta^\diamond; a_\delta^\diamond, b_\delta^\diamond)$ be a family of Dobrushin domains converging to a Dobrushin domain $(\Omega; a, b)$ in the Carathéodory sense. The interface of the critical Ising model in $(\Omega_\delta^*, a_\delta^*, b_\delta^*)$ with Dobrushin boundary condition converges weakly to SLE_3 as $\delta \rightarrow 0$.*

5.2 Proof of Theorem 1.3

Let $(\Omega_\delta; x_\delta^L, x_\delta^R, y_\delta^R, y_\delta^L)$ be a sequence of discrete quads on the square lattice $\delta\mathbb{Z}^2$ approximating some quad $(\Omega; x^L, x^R, y^R, y^L)$. Consider the critical Ising model in Ω_δ^* with alternating boundary conditions: \ominus along $(x_\delta^L x_\delta^R)$ and $(y_\delta^R y_\delta^L)$, and \oplus along $(x_\delta^R y_\delta^R)$ and $(y_\delta^L x_\delta^L)$. Suppose there is a vertical crossing of \ominus and denote this event by

$$\mathcal{C}_v^\ominus(\Omega_\delta) = \{(x_\delta^L x_\delta^R) \xleftrightarrow{\ominus} (y_\delta^R y_\delta^L)\}.$$

Let η_δ^L be the interface starting from x_δ^L lying on the primal lattice. It turns at every vertex in the way that it has spin \oplus on its left and \ominus on its right, and that it turns left when there is ambiguity. Let η_δ^R be the interface starting from x_δ^R lying on the primal lattice. It turns at every vertex in the way that it has spin \ominus to its left and \oplus to its right, and turns right when there is ambiguity. Then η_δ^L will end at y_δ^L and η_δ^R will end at y_δ^R . See Figure 1.1. Let Ω_δ^L be the connected component of $\Omega_\delta \setminus \eta_\delta^L$ with $(x_\delta^R y_\delta^R)$ on the boundary and denote by \mathcal{D}_δ^L the discrete extremal distance between η_δ^L and $(x_\delta^R y_\delta^R)$ in Ω_δ^L . Define Ω_δ^R and \mathcal{D}_δ^R similarly.

Lemma 5.3. *The variables $(\mathcal{D}_\delta^L; \mathcal{D}_\delta^R)_{\delta>0}$ is tight in the following sense: for any $u > 0$, there exists $\epsilon > 0$ such that*

$$\mathbb{P}[\mathcal{D}_\delta^L \geq \epsilon, \mathcal{D}_\delta^R \geq \epsilon \mid \mathcal{C}_v^\ominus(\Omega_\delta)] \geq 1 - u, \quad \forall \delta > 0.$$

Proof. Since $(\Omega_\delta; x_\delta^L, x_\delta^R, y_\delta^R, y_\delta^L)$ approximates $(\Omega; x^L, x^R, y^R, y^L)$, by Proposition 5.1 and (5.1), we know that $\mathbb{P}[\mathcal{C}_v^\ominus(\Omega_\delta)]$ can be bounded from below by some quantity that depends only on the extremal distance in Ω between $(x^L x^R)$ and $(y^R y^L)$ and that is uniform over δ . Thus, it is sufficient to show $\mathbb{P}[\{\mathcal{D}_\delta^L \leq \epsilon\} \cap \mathcal{C}_v^\ominus(\Omega_\delta)]$ is small for $\epsilon > 0$ small. Given η_δ^L and on the event $\{\mathcal{D}_\delta^L \leq \epsilon\}$, we know that the configuration in Ω_δ^L is critical Ising model with Dobrushin boundary condition. Combining Proposition 5.1 and (5.1), we know that the probability to have a vertical crossing of \ominus in Ω_δ^L is bounded by $c(\epsilon)$ which only depends on ϵ and goes to zero as $\epsilon \rightarrow 0$. Thus $\mathbb{P}[\{\mathcal{D}_\delta^L \leq \epsilon\} \cap \mathcal{C}_v^\ominus(\Omega_\delta)] \leq c(\epsilon)$. This implies the conclusion. \square

Proof of Theorem 1.3. Combining Proposition 5.1 with (5.1) and Lemma 5.3, we see that the sequence $\{(\eta_\delta^L; \eta_\delta^R)\}_{\delta>0}$ satisfies the requirements in Theorem 2.4, thus the sequence is relatively compact. Suppose $(\eta^L; \eta^R) \in X_0(\Omega; x^L, x^R, y^R, y^L)$ is any sub-sequential limit and, for some $\delta_k \rightarrow 0$,

$$(\eta_{\delta_k}^L; \eta_{\delta_k}^R) \xrightarrow{d} (\eta^L; \eta^R) \quad \text{in } X_0(\Omega; x^L, x^R, y^R, y^L).$$

Since $\eta_{\delta_k}^L \rightarrow \eta^L$, by Theorem 2.2, we know that we have the convergence in all three topologies. In particular, this implies the convergence of $\Omega_{\delta_k}^L$ in Carathéodory sense. Note that the conditional law of $\eta_{\delta_k}^R$ in $\Omega_{\delta_k}^L$ given $\eta_{\delta_k}^L$ is the interface of the critical planar Ising model with Dobrushin boundary condition. Combining with Theorem 5.2, we know that, the conditional law of η^R in Ω^L given η^L is SLE_3 . By symmetry, the conditional law of η^L in Ω^R given η^R is SLE_3 . By Theorem 1.2, there exists a unique such measure. Thus it has to be the unique sub-sequential limit. This completes the proof of Theorem 1.3. \square

6 FK-Ising Model and Proof of Proposition 1.5

6.1 FK-Ising model

We will consider finite subgraphs $G = (V(G), E(G)) \subset \mathbb{Z}^2$. For such a graph, we denote by ∂G the inner boundary of G :

$$\partial G = \{x \in V(G) : \exists y \notin V(G) \text{ such that } \{x, y\} \in E(\mathbb{Z}^2)\}.$$

A *configuration* $\omega = (\omega_e : e \in E(G))$ is an element of $\{0, 1\}^{E(G)}$. If $\omega_e = 1$, the edge e is said to be open, otherwise e is said to be closed. The configuration ω can be seen as a subgraph of G with the same set of vertices $V(G)$, and the set of edges given by open edges $\{e \in E(G) : \omega_e = 1\}$.

We are interested in the connectivity properties of the graph ω . The maximal connected components of ω are called clusters. Two vertices x and y are connected by ω inside $S \subset \mathbb{Z}^2$ if there exists a path of vertices $(v_i)_{0 \leq i \leq k}$ in S such that $v_0 = x, v_k = y$ and $\{v_i, v_{i+1}\}$ is open in ω for $0 \leq i < k$. We denote this event by $\{x \xrightarrow{S} y\}$. If $S = G$, we simply drop it from the notation. For $A, B \subset \mathbb{Z}^2$, set $\{A \xrightarrow{S} B\}$ if there exists a vertex of A connected in S to a vertex in B .

Given a finite subgraph $G \subset \mathbb{Z}^2$, *boundary condition* ξ is a partition $P_1 \sqcup \dots \sqcup P_k$ of ∂G . Two vertices are wired in ξ if they belong to the same P_i . The graph obtained from the configuration ω by identifying the wired vertices together in ξ is denoted by ω^ξ . Boundary conditions should be understood informally as encoding how sites are connected outside of G . Let $o(\omega)$ and $c(\omega)$ denote the number of open and dual edges of ω and $k(\omega^\xi)$ denote the number of maximal connected components of the graph ω^ξ .

The probability measure $\phi_{p,q,G}^\xi$ of the *random cluster model* on G with edge-weight $p \in [0, 1]$, cluster-weight $q > 0$ and boundary condition ξ is defined by

$$\phi_{p,q,G}^\xi[\omega] := p^{o(\omega)} (1-p)^{c(\omega)} q^{k(\omega^\xi)} / Z_{p,q,G}^\xi,$$

where $Z_{p,q,G}^\xi$ is the normalizing constant to make $\phi_{p,q,G}^\xi$ a probability measure. For $q = 1$, this model is simply Bernoulli bond percolation.

For a configuration ξ on $E(\mathbb{Z}^2) \setminus E(G)$, the boundary conditions induced by ξ are defined by the partition $P_1 \sqcup \dots \sqcup P_k$, where x and y are in the same P_i if and only if there exists an open path in ξ connecting x and y . We identify the boundary condition induced by ξ with the configuration itself, and denote the random cluster model with these boundary conditions by $\phi_{p,q,G}^\xi$. As a direct consequence of these definitions, we have the Domain Markov Property of the random cluster model: Suppose that $G' \subset G$ are two finite subgraphs of \mathbb{Z}^2 . Fix $p \in [0, 1], q > 0$ and ξ some boundary conditions on ∂G . Let X be a random variable which is measurable with respect to edges in $E(G')$. Then we have

$$\phi_{p,q,G}^\xi[X \mid \omega_e = \psi_e, \forall e \in E(G) \setminus E(G')] = \phi_{p,q,G}^{\psi^\xi}[X], \quad \forall \psi \in \{0, 1\}^{E(G) \setminus E(G')},$$

where ψ^ξ is the partition on $\partial G'$ obtained as follows: two vertices $x, y \in \partial G'$ are wired if they are connected in ψ^ξ .

Suppose that $(\Omega; a, b)$ is a Dobrushin domain. The Dobrushin boundary condition is the following: free along (ab) and wired along (ba) . Suppose that $(\Omega; a, b, c, d)$ is a quad. The alternating boundary condition is the following: free along (ab) and (cd) , wired along (bc) and (da) .

Denote the product ordering on $\{0, 1\}^E$ by \leq . In other words, for $\omega, \omega' \in \{0, 1\}^E$, we denote by $\omega \leq \omega'$ if $\omega_e \leq \omega'_e$, for all $e \in E$. An event \mathcal{A} depending on edges in E is *increasing* if for any $\omega \in \mathcal{A}, \omega \leq \omega'$ implies $\omega' \in \mathcal{A}$. We have positive association (FKG inequality) when $q \geq 1$: Fix $p \in [0, 1], q \geq 1$ and a finite graph G and some boundary conditions ξ . For any two increasing events \mathcal{A} and \mathcal{B} , we have

$$\phi_{p,q,G}^\xi[\mathcal{A} \cap \mathcal{B}] \geq \phi_{p,q,G}^\xi[\mathcal{A}] \phi_{p,q,G}^\xi[\mathcal{B}].$$

As a consequence of the FKG inequality, we have the comparison principle between boundary conditions: fix $p \in [0, 1], q \geq 1$ and a finite graph G . For any boundary conditions $\xi \leq \psi$ and any increasing event \mathcal{A} , we have

$$\phi_{p,q,G}^\xi[\mathcal{A}] \leq \phi_{p,q,G}^\psi[\mathcal{A}].$$

For a finite subgraph G , we define G^* to be the subgraph of $(\mathbb{Z}^2)^*$ with edge-set $E(G^*) = \{e^* : e \in E(G)\}$ and vertex set given by the end-points of these dual-edges. A configuration ω on G can be uniquely associated to a dual configuration ω^* on the dual graph G^* defined as follows: set $\omega^*(e^*) = 1 - \omega(e)$ for all $e \in E(G)$. A dual-edge e^* is said to be dual-open if $\omega^*(e^*) = 1$, it is dual-closed otherwise. A dual-cluster is a connected component of ω^* . We extend the notion of dual-open path and the connective events in the obvious way. If ω is distributed according to $\phi_{p,q,G}^\xi$, then ω^* is distributed according to $\phi_{p^*,q^*,G^*}^{\xi^*}$ where

$$q^* = q, \quad \frac{pp^*}{(1-p)(1-p^*)} = q,$$

and the boundary conditions ξ^* can be deduced from ξ in a case by case manner. In particular, $\xi = 0$ corresponds to $\xi^* = 1$ and $\xi = 1$ corresponds to $\xi^* = 0$.

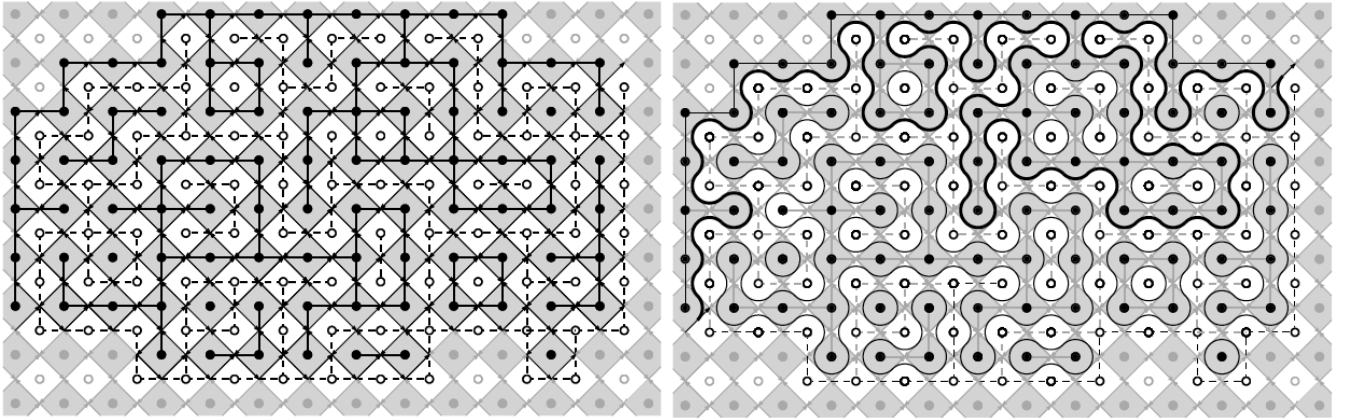
The critical value of p for a given q is the following:

$$p_c(q) = \frac{\sqrt{q}}{1 + \sqrt{q}}.$$

People believe that the critical random-cluster model is conformal invariant in the scaling limit for $q \in [1, 4]$, and it is only proved for $q = 2$ in [CS12, CDCH⁺14]. When $q = 2$, the critical random-cluster model is also called FK-Ising model. We will list two special properties for the FK-Ising model that will be useful later: strong RSW and the convergence of the interface with Dobrushin boundary condition.

Proposition 6.1. [CDCH16, Theorem 1.1]. *For each $L > 0$ there exists $c(L) > 0$ such that the following holds: for any quad (Q, a, b, c, d) with $d_Q((ab), (cd)) \geq L$ and for any boundary conditions ξ ,*

$$\phi_{p_c(2), 2, \Omega}^\xi [(ab) \leftrightarrow (cd)] \leq 1 - c(L).$$



(a) The configuration ω and its dual ω^* .

(b) The loop representation of ω .

Fig. 6.1: The loop representation of the configuration.

Fix a Dobrushin domain $(\Omega; a, b)$ and consider a configuration ω together with its dual-configuration ω^* . The Dobrushin boundary condition is given by taking edges of ∂_{ba} to be open and the dual-edges of ∂_{ab}^* to be dual-open. Through every vertex of Ω° , there passes either an open edge of Ω or a dual open edge of Ω^* . Draw self-avoiding loops on Ω° as follows: a loop arriving at a vertex of the medial lattice always makes a $\pm\pi/2$ turn so as not to cross the open or dual open edges through this vertex, see Figure 6.1. The loop representation contains loops together with a self-avoiding path going from a° to b° . This curve is called the *exploration path*. See [DC13, Section 6.1] for more details.

Theorem 6.2. [CDCH⁺14]. *Let $(\Omega_\delta; a_\delta, b_\delta)$ be a family of Dobrushin domains converging to a Dobrushin domain $(\Omega; a, b)$ in the Carathéodory sense. The exploration path of the critical FK-Ising model with Dobrushin boundary conditions converges weakly to $\text{SLE}_{16/3}$ as $\delta \rightarrow 0$.*

6.2 Proof of Proposition 1.5

Lemma 6.3. *Let $(\Omega_\delta; a_\delta, b_\delta, c_\delta, d_\delta)$ be a family of quads converging to a quad $(\Omega; a, b, c, d)$ in the Carathéodory sense. Consider the critical FK-Ising model with Dobrushin boundary condition: wired on $(d_\delta a_\delta)$ and free*

on $(a_\delta d_\delta)$ and denote by η_δ the exploration path from a_δ to d_δ . Denote by $\mathcal{C}_v^0(\Omega_\delta)$ the event that there is a dual-crossing in Ω_δ from $(a_\delta b_\delta)$ to $(c_\delta d_\delta)$. Then the law of η_δ conditioned on the event $\mathcal{C}_v^0(\Omega_\delta)$ converges weakly to $\text{SLE}_{16/3}$ in Ω from a to d conditioned not to hit (bc) as $\delta \rightarrow 0$. Given η_δ and on the event $\mathcal{C}_v^0(\Omega_\delta)$, define \mathcal{D}_δ to be the discrete extremal distance between η_δ and $(b_\delta c_\delta)$ in Ω_δ^η which is the connected component of $\Omega_\delta \setminus \eta_\delta$ with $(b_\delta c_\delta)$ on the boundary. Then $\{\mathcal{D}_\delta\}_{\delta>0}$ is tight in the following sense: for any $u > 0$, there exists $\epsilon > 0$ such that

$$\mathbb{P}[\mathcal{D}_\delta \geq \epsilon] \geq 1 - u, \quad \forall \delta > 0.$$

Proof. The convergence is a direct consequence of Theorem 6.2. The tightness can be proved by the same argument as in the proof of Lemma 5.3 where we need to replace Proposition 5.1 by Proposition 6.1. \square

Lemma 6.4. *Let $(\Omega_\delta; x_\delta^L, x_\delta^R, y_\delta^R, y_\delta^L)$ be a sequence of discrete quads on the square lattice $\delta\mathbb{Z}^2$ approximating some quad $(\Omega; x^L, x^R, y^R, y^L)$. Consider the critical FK-Ising model in Ω_δ with alternating boundary conditions: free on $(x_\delta^L x_\delta^R)$ and $(y_\delta^R y_\delta^L)$, and wired on $(x_\delta^R y_\delta^R)$ and $(y_\delta^L x_\delta^L)$. Conditioned on the event that there are two disjoint vertical dual-crossings, then there exists a pair of interfaces $(\eta_\delta^L; \eta_\delta^R)$ where η_δ^L (resp. η_δ^R) is the interface connecting x_δ^L to y_δ^L (resp. connecting x_δ^R to y_δ^R). The law of $(\eta_\delta^L; \eta_\delta^R)$ converges weakly to the unique pair of curves $(\eta^L; \eta^R)$ in $X_0(\Omega; x^L, x^R, y^R, y^L)$ with the following property: Given η^L , the conditional law of η^R is an $\text{SLE}_{16/3}$ conditioned not to hit η^L ; given η^R , the conditional law of η^L is an $\text{SLE}_{16/3}$ conditioned not to hit η^R .*

Proof. Combining Proposition 6.1 and Lemma 6.3, we see that the sequence $\{(\eta_\delta^L; \eta_\delta^R)\}_{\delta>0}$ satisfies the requirements in Theorem 2.4, thus the sequence is relatively compact. Suppose $(\eta^L; \eta^R)$ is any subsequential limit and, for some $\delta_k \rightarrow 0$, we have $(\eta_{\delta_k}^L; \eta_{\delta_k}^R) \rightarrow (\eta^L; \eta^R)$ in distribution. Let $\Omega_{\delta_k}^L$ be the connected component of $\Omega \setminus \eta_{\delta_k}^L$ with $(x_{\delta_k}^R y_{\delta_k}^R)$ on the boundary, and define Ω^L similarly. Since $\eta_{\delta_k}^L \rightarrow \eta^L$, combining with Theorem 2.2, we have the convergence in all three topologies. In particular, the quads $\Omega_{\delta_k}^L$ converges to Ω^L in Carathéodory sense. Combining with Lemma 6.3, we know that the conditional law of η^R given η^L is $\text{SLE}_{16/3}$ conditioned not to hit η^L . By symmetry, the conditional of η^L given η^R is $\text{SLE}_{16/3}$ conditioned not to hit η^R . It remains to explain that there is a unique measure on pairs $(\eta^L; \eta^R)$ with such property. The uniqueness can be proved by the same argument as in the proof of Proposition 4.1. The key point is that the two curves η^L, η^R do not hit each other and that the counterflow line is deterministic function of the GFF and thus the event that the counterflow line does not hit part of the boundary is also deterministic of the GFF. \square

Proof of Proposition 1.5. Fix $\kappa = 16/3$. We derive the conclusion by three steps: first, let the quads $(\Omega_\delta; x_\delta^L, x_\delta^R, y_\delta^R, y_\delta^L)$ approximate some quad $(\Omega; x^L, x^R, y^R, y^L)$; second, let $y^L, y^R \rightarrow y$; thirdly, let $x^L, x^R \rightarrow x$. In the first step, by Lemma 6.4, we know that the pairs of interfaces converge weakly to a unique probability measure on $(\eta^L; \eta^R) \in X_0(\Omega; x^L, x^R, y^R, y^L)$ such that the conditional law of η^R given η^L is SLE_κ conditioned not to hit η^L and the conditional law of η^L given η^R is SLE_κ conditioned not to hit η^R . Now, let us fix η^L and let $y^R \rightarrow y := y^L$. By Lemma 2.7, we know that the conditional law of η^R given η^L converges weakly to $\text{SLE}_\kappa(\kappa - 4)$. Let $X_0^2(x^L, x^R, y)$ be the collection of pairs of curves $(\eta^L; \eta^R)$ such that $\eta^L \in X_0(\Omega; x^L, y), \eta^R \in X_0(\Omega; x^R, y)$ and that η^L is to the left of η^R . From the above analysis, by sending $y^L, y^R \rightarrow y$, the limiting probability measure on $(\eta^L; \eta^R) \in X_0^2(x^L, x^R, y)$ satisfies the following property: the conditional law of η^L given η^R is $\text{SLE}_\kappa(\kappa - 4)$ and the conditional law of η^R given η^L is $\text{SLE}_\kappa(\kappa - 4)$. By Lemma 4.5, there exists a unique such measure, and under this measure, the marginal law of η^L is $\text{SLE}_\kappa(\kappa - 2)$ with force point located at x^R and the marginal law of η^R is $\text{SLE}_\kappa(\kappa - 2)$ with force point located at x^L . Finally, since the law of $\text{SLE}_\kappa(\rho)$ process is continuous in the locations of the force points, we complete the proof by sending $x^L, x^R \rightarrow x$. \square

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